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RELATIVITY

A SYSTEMATIC TREATMENT OF EINSTEIN'S THEORY

BY

J. RICE, M.A.

SENIOR LECTURER IN PHYSICS AT THE UNIVERSITY OF LIVERPOOL

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PREFACE.

WITHIN the past twenty years a revolution has taken place in the mental attitude which the physicist maintains towards the concepts which have been invented in the past in order to reach those broad generalisations which are the proud possession of his science.

The demand on the part of the layman to know just what this revolution portends has been satisfied, as far as may be, by a liberal supply of popular works on Relativity. But the science undergraduate taking his normal courses in the University classroom, or reading his text-books of Physics and Mathematics, is anxious to ascertain in a more precise manner what changes this new idea is producing in the principles and content of physical science. It is primarily for him that this book has been written. Unfortunately it is only too easy to acquire the notion that the new knowledge is dealing out death and destruction to the principles won so laboriously since the time of Galileo and Newton. To correct such a disastrous misapprehension it is highly desirable that the mind should be gradually adapted to the new idea as the usual University courses are pursued, and not be compelled suddenly to readjust its point of view after the work involved in obtaining a degree has been completed. In short, Relativity must not remain something whose full significance can only be grasped by a small band of highly-trained specialists, but must be regarded as a selective principle with which the young student should make as early an acquaintance as possible.

Since the undergraduate has been familiarised in

his school and first year courses with the elements of dynamical science, this book begins, after a general Introduction, with a discussion of the modifications in Kinematics and Dynamics required by the acceptance of the Relativity standpoint. Historically, of course, the new theory grew out of the theory of the electromagnetic field; but as the treatment of the field by means of Maxwell's equations is a branch of physical science which comes comparatively late in the usual courses, the relativity of the field equations is discussed after consideration of the equations of Dynamics.* In these earlier chapters, the mathematical method used is familiar and well within the student's powers.

In Chapter V. a systematic development of the Tensor Analysis, which has proved to be the suitable mathematical medium for the application of the Relativity test, is initiated, and the remaining chapters of Part I. deal with the discovery by its aid of those forms for the differential equations of dynamical and electromagnetic theory which are compatible with the restrictions imposed on the transformations of coordinates in Einstein's first statement of the Relativity principle in 1905. The abandonment of these restrictions, leading to the introduction of the principle of Equivalence and Einstein's law of gravitation, and the requisite generalisation of the mathematical method are treated in Part II. Some care has been taken to indicate precisely the points of similarity between the Tensor Analysis used in Part I. and the Tensor Analysis for transformations with variable coefficients employed in Part II. Chapters X. and XI. are to some extent parallel treatments of the matters discussed in Chapters VI. and VII. with the restrictions of the earlier statement of the Principle removed. Chapter XII. is devoted to a discussion of the principle of Stationary Action, not only with regard to the important place which it

* The appearance of Relativity as a definite subject in the curricula, may involve as a necessary change an earlier appearance than hitherto of Maxwell's equations in Physics courses, a change not undesirable from other points of view.

attained in the Physics of the nineteenth century, but also in connection with the possibility of its maintenance of that position in the new synthesis. A chapter on the purely mathematical problem of solving Einstein's gravitational equation with the bearings of the solutions on the questions of planetary orbits, deviation of light beams, gravitational displacement of spectral lines and gravitational waves, brings Part II. to a close.

Doubtless the reading of these two parts will involve as much time and thought as even the most industrious student can spare from the demands of the other branches of the physical and mathematical sciences in his undergraduate years. But no book on Relativity published at present would be complete without some account of the very interesting developments which have taken place since 1917. Not only for the sake of post-graduate reading, but in order to appeal to a wider circle of readers, a fairly complete account is given in Part III. of the cosmological speculations of Einstein and de Sitter and the attempts by Weyl, Eddington and Einstein to derive a mathematical theory of the electromagnetic field (as well as of the gravitational) from the treatment of the metric field of space-time. In order to follow these researches some information concerning Riemann's analysis of a manifold which can be represented in a multidimensional space is required. This is supplied in Chapter XIV. In Chapter XV. the question of the finiteness of our universe and its bearing on the boundary conditions to be imposed on the differential equations of Physics is discussed. Chapter XVI. refers in some detail to the remarkable analogies pointed out by Eddington between the tensors which can be derived from the "curvature tensor" of Weyl's generalised geometry of a manifold and the tensors representing the important concepts of present-day physical science. It has just been found possible to incorporate in the last pages a summary of the most recent contribution by Einstein, a paper which may prove to be of cardinal importance in future developments.

The book is confined entirely to the physical and mathematical aspects of the theory, and makes no pretence to enter into any of the discussions which it has provoked in philosophical circles. The author has, of course, drawn on all the published sources of information known to him, but of the books published on the subject he is specially indebted to the early works of Dr. Silberstein and Dr. Cunningham on the Restricted Principle, to Professor Eddington's "Report" and to his "Space, Time and Gravitation," one of the literary gems of English scientific literature. Reference has also been made to certain parts of Weyl's "Raum, Zeit, Materie," of which an English translation has recently appeared. The treatment of the relation of electron theory to the question of a finite universe, given in Chapter XV., is taken from "Le Principe de Relativité," by M. Jean Becquerel.

The author wishes to thank Dr. E. L. Ince of the Mathematical Department in this University for his assistance in reading proof-sheets, and for many valuable suggestions and criticisms.

THE UNIVERSITY OF LIVERPOOL,
May, 1923.

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The natural path of a free particle in a specified frame is given by the geodesic equation. This statement forms the starting-point of the new dynamics, and in it the ideas of inertia and gravitation are united. Some criterion has to be set up which distinguishes between natural gravitational fields and the geometric fields, which can be derived by an arbitrary transformation of co-ordinates. This criterion is given by Einstein's "Law of Gravitation" in the gravitational field (as distinct from matter). The law degenerates into Laplace's equation under certain restrictive conditions. The attitude of the relativist towards co-ordinates introduces a certain element of doubt and perplexity in the interpretation of analytical results, which, however, can be largely removed by the consideration of degenerate cases. The law of gravitation limits the principle of Equivalence. Possible simplification of Einstein's equation by a suitable choice of co-ordinates . . *page* 190

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In his *geometry* the ambiguity involved in preserving the direction of a vector transported in a curved space (already treated in Chapter XIV.) is accompanied by a further ambiguity in preserving the magnitude of the vector as well. Questions of magnitude must be settled in practice by the use of material measuring standards, and Weyl assumes as a *physical* possibility (quite apart from formal geometrical reasoning) that the transportation of a material standard from place to place may affect its size in a manner *which is dependent on the route chosen for carrying it between the two places.*

The *geometrical* generalisation involves the introduction of four gauge coefficients, forming a covariant vector. The *physical* assumption is made that the postulated change of size in the material standard depends on the electromagnetic field through which the route of transference passes. Weyl now proceeds to a synthesis in which the four geometrical gauge-coefficients are identified with the four physical quantities, the components of the covariant electromagnetic potential. This generalisation would render Einstein's previous theory approximate, and only exact in the absence of fields of electromagnetic force; for in order to preserve complete relativity of the laws of nature, we should have to invent a new type of tensor, which is independent not only of transformation of co-ordinates but also of arbitrary changes of gauges. Such tensors and invariants, "in-tensors" and "in-invariants," can be found, and if Weyl's *physical* theory be upheld the equations expressing the laws of nature would have to be formulated in terms of these. However, Eddington has pointed out that this rather drastic revision of the theory is unnecessary. While maintaining Weyl's wider treatment of geometry, he shows that his physical assumption as to changes in material standards is uncalled-for Actually there is a privileged gauge which we can use in comparisons of size at different places and in different directions. The gauge at a specified place in a specified direction is the radius of curvature of the universe at that place and in that direction. Eddington further shows that the in-tensor of the second order, which we arrive at by the treatment of parallel displacement from the wider geometrical aspect, gives us by a natural resolution three tensors, one of which can be identified with the original Einstein tensor, the second with the field tensor of the electromagnetic field, and the third possibly with the as yet unknown structure of the electron. The *method of identification* thus introduced is further developed with regard to the principal quantities involved in Physical Science, and it appears that if we adopt the concept of parallel displacement rather than the idea of metrical separation as a geometrical starting-point, we can place all physical forces on an equal footing in the Relativity Theory and maintain Einstein's equations as exact and not approximate. The difficulties of the electron theory are referred to. The possibility of introducing certain natural units for physical quantities leads to a natural unit of action which may be related to the quantum of action involved in Planck's theory . . . . . *page* 349

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NOTE ON THE MATHEMATICAL NOTATION.

IN order to economise space and prevent the occurrence of broken lines and irregular spacing, the solidus is employed instead of the more customary horizontal stroke in printing fractional expressions and differential coefficients. Thus expressions such as

$$\frac{u}{c}, \quad \frac{v}{(1 - v^2)^{\frac{1}{2}}}, \quad \frac{c^6}{g^2 x^3}$$

are printed as

$$u/c, \quad v/(1 - v^2)^{\frac{1}{2}}, \quad c^6/g^2x^3,$$

although in many cases the use of negative indices is resorted to. Differential coefficients such as

$$\frac{d\phi}{dx}, \quad \frac{d(mv)}{dt}, \quad \frac{\partial(g^{\mu\beta}\{\lambda\beta, \alpha\})}{\partial x_\alpha}$$

are printed as

$$d\phi/dx, \quad d(mv)/dt, \quad \partial(g^{\mu\beta}\{\lambda\beta, \alpha\})/\partial x_\alpha.$$

Wherever it is necessary to indicate the product of such expressions, a dot is printed between. Thus

$$\frac{\partial\phi}{\partial x_\alpha}\frac{dx_\alpha}{ds}$$

is printed as

$$\partial\phi/\partial x_\alpha \cdot dx_\alpha/ds.$$

lation of optical and electromagnetic theory during the nineteenth century revived the hope of discovering the earth's absolute motion in space by means of the laws derived from this theory. One direct method which was suggested, consisted in an endeavour to measure differences in the speeds of light in different directions. It was argued that light would have a definite and unique velocity through space; but that the velocity through terrestrial apparatus would be this velocity increased or decreased by an amount depending on the earth's velocity through space and its relative direction to that of the source emitting the light, so that if the earth were moving towards the source the measured velocity would be equal to the real velocity of light plus that of the earth, while if the earth were moving away from the source the plus would be read as minus; just as two trains pass each other with a relative speed equal to the sum of their track speeds if they are travelling in opposite directions. This method seemed so hopeful that a great deal of patience and skill was expended on perfecting apparatus for this purpose, and in the end the American physicist Michelson and his pupils devised an instrument so ingenious and precise that failure to measure the anticipated differences was inconceivable—provided they existed. Yet failure followed each successive attempt despite every improvement in apparatus and technique. No charge could be brought against the skill and precision of the experimentalist; the only conclusion was that somewhere in the reasoning of the theoretician there was an erroneous assumption.

Michelson's experiment consisted in timing a race between two beams of light, each of which, starting from a common point A, travelled to reflecting mirrors and returned to A. Two mirrors, B and C, were set so that AB and AC were at right angles. Supposing the apparatus to be at rest in absolute space, the time of each journey would be $2l_1/c$ and $2l_2/c$, where $AB = l_1$, $AC = l_2$, and the velocity of light in space is represented by c. But the apparatus was, of course, situated on the earth, and so was in movement through space with a velocity, which during the time occupied by an experiment, could be considered as uniform. Represent this velocity by u Then the expressions written above would no longer represent the times involved in the race, if the considerations advanced in the previous paragraph be valid. Suppose, for example, that AB were parallel to the direction of u and AC therefore perpendicular to u, it can be shown by elementary

kinematical reasoning that the beam travelling to B and back would take the time $2a^2l_1/c$, and the other would take $2al_2/c$, where a is written as a convenient symbol for the expression

$$(1 - u^2/c^2)^{-\frac{1}{2}}$$

and is, of course, always greater than unity. Hence the difference in the times occupied by the two journeys would be $2(a^2l_1 - al_2)/c$. Were AB, however, perpendicular to the direction of u and AC parallel to it, this difference would be $2(al_1 - a^2l_2)/c$. These results for the two particular settings of the apparatus will account for the expectation of the experimenters that a gradual rotation of their apparatus should have revealed a change in the time by which one of the beams arrived at A " ahead " of the other. The optical method employed to detect the anticipated change was certainly sensitive enough, even if the velocity of the earth through space were only of the order of its known relative velocity in its solar orbit. Yet not the slightest alteration in this difference could be detected despite experimental refinements and precautions which left no doubt that the absence of the expected effect was not due to lack of precision in the apparatus. Suggestions for evading the difficulty took the line of assuming either that the ether round the earth was not at rest, but was carried along with it, or else that a source of light in motion imparted its velocity to the light and produced, when compounded with the " true " velocity of light, a resultant velocity whose magnitude depended on the velocity of the source and on the relative direction of the light beam and the motion of the source.

Suffice it to say that neither of these suggestions survived the criticism of physicist and astronomer, and in a short time physicists came to accept a totally different suggestion first propounded by Fitzgerald, but worked out independently and much more completely by Lorentz. This amounted to an assumption that the dimensions of a body alter when it is in motion through the ether, as compared with its dimensions when at rest; that in fact a length of material equal to l, when at rest, would contract to a length l/a or $l(1 - u^2/c^2)^{\frac{1}{2}}$ when moving with a velocity u through the ether in a direction parallel to this length, c being the velocity of light through the ether; across the direction of motion no alteration in size takes place. This contraction being independent of the material of the body and only depending on its velocity, it could not be detected by measurement with any material

standard of length; for when body and standard were juxtaposed for comparison, each would suffer exactly the same change, because each would have the same velocity through absolute space, and so coincidence of marks on the body with definite graduations on the standard would not be disturbed. In fact, the hypothesis of contraction stood by itself, unrelated to any other physical fact and merely accepted because it enabled physicists to maintain the hypothesis of the stagnant, immobile, and universal ether in face of Michelson's negative result. How it achieved this is easily grasped by reference to the expressions above. Thus, when AB is parallel to u and AC perpendicular to it, the length of AB is not l_1, but l_1/a, while that of AC is still l_2. Hence, the times for the two beams are $2a^2l_1/ac$ and $2al_2/c$, which give as the difference $2a(l_1 - l_2)/c$, and in the other orientation the times are $2al_1/c$ and $2a^2l_2/ac$, yielding just the same difference, viz., $2a(l_1 - l_2)/c$.

Despite the satisfaction derived from a hypothesis which so neatly explained the Michelson experiment, it still suffered from the defect mentioned above, viz., its lack of relation to any other physical change in the presumably contracted body. It is known, for instance, that ordinary mechanical strains in transparent, isotropic materials like glass would change their optical behaviour, endowing them with the property of double refraction, for example. The idea occurred to Rayleigh that although the contractions due to terrestrial motion through the ether would evade detection by direct measurement, yet they would render a body doubly refracting with its optic axis along the direction of motion, so that if the body were rotated the change in the orientation of the optic axis in the body itself could be detected. He instituted a search for this phenomenon, but in vain. A second research by Brace with more powerful experimental appliances proved equally futile. Other experiments, seeking with every hope of success to discover some physical alteration in the contracted body (e.g., a variation in the electrical resistance of a wire) as its orientation was changed, met with no better fate. Purely negative results were the invariable outcome of these attempts. Baffled in their search for this absolute motion of the earth, physicists were compelled to review the whole foundations of their science, and especially that part of it dealing with electromagnetic phenomena. It was just at this period that the theory of the electrical constitution of matter was being constructed and giving evidence of its great power; and it was at once recognised that this theory would have to be framed so as to account for the fact that all

the properties of a portion of matter were so nicely modified by any change in its motion through the ether as just to obliterate the effects which might naturally be expected to show themselves on account of this change of motion. On the one hand, it was discovered that the Lorentz-Fitzgerald contraction had to be not merely a property of matter in bulk, but also an effect experienced by the most minute portions of matter, electrons and atomic nuclei. On the other, it was recognised that Maxwell's equations of the electromagnetic field would have to be valid not only for axes of reference fixed in the ether, but also valid *in the same mathematical form* for axes moving uniformly through the ether. Now this point, which is vital to the true understanding of Relativity, is one which is extremely difficult for the non-mathematical reader to grasp. As we have just said, we are committed to an electrical theory of matter, and this theory is built on the validity of Maxwell's equations. We postulate the existence of two physical vectors, an electric and a magnetic intensity, which have definite values at every point, and Maxwell's equations indicate to those who are conversant with mathematical symbolism, the manner in which a knowledge of the values *at one moment* of these vectors at all the points surrounding, and in the immediate neighbourhood of, a definite point can lead to the values of these vectors at this point at a later moment. The manner in which the value of either vector at a given point varies during a brief interval of time is quantitatively determined by the differences which exist at one instant between the value of the other vector at that point and its values at the points within a small region surrounding the given one. The equations are, in fact, a highly symbolic guide for predicting the history of the vectors. If these equations have what we may call their standard form in a system of axes fixed in the ether, it would not be unnatural to expect that if we use co-ordinate axes attached to a piece of matter in motion through the ether, the form of the equations suitable for such axes could not be the same as before, but would involve a symbol, u, which would represent the speed of the matter through the ether. But if this expectation is satisfied, it is not difficult for the mathematician to show that it can be deduced from the equations that the velocity of light relative to the moving axes will depend on u, and, further, that the experiments mentioned above should yield positive results. So, in order to agree with known facts, the equations for the moving frame of reference must not contain u, and they must have the same mathematical form as before. If the moving

frame is travelling parallel to a common axis of x of the two systems of axes, the co-ordinates of a definite point in the ether would be e.g. (x, y, z), referred to the fixed system, while referred to the moving system they would at time t, be

$$\left.\begin{array}{l} x' = x - ut \\ y' = y\,;\;\; z' = z \end{array}\right\} \quad . \quad . \quad . \quad (1)$$

Electromagnetic theory shows that the values of the electric and magnetic vectors at this point as observed by the fixed experimenter would differ from the values observed by the moving one; these can be represented by E and H for the former, and by E′ and H′ for the latter, and known mathematical relations are available for determining E′ and H′ in terms of E, H, and u. It is not necessary at the moment to inquire what these equations should be, for it can be shown that if we retain equations (1), which are a direct expression of Newtonian kinematics, then it is impossible to derive from Maxwell's equations in the symbols E, H. x, y, z, t equations *of the same mathematical form* in the symbols E′, H′, x', y', z', t. But this is apparently what we must do to satisfy the experimental results. Lorentz was able to show that we could go some way towards the desired goal if instead of (1) we substituted

$$\left.\begin{array}{l} x' = a(x - ut) \\ y' = y\,;\;\; z' = z \end{array}\right\} \quad . \quad . \quad . \quad . \quad (2)$$

a being, as before, a short symbol for $(1 - u^2/c^2)^{-\frac{1}{2}}$. This change after all was quite in keeping with the development of ideas since it was just the analytical change required to allow for the postulated contraction of all the standards of measurement used by the moving observers. Nevertheless, it was insufficient for the purpose in view. At last Lorentz was able to show that if we introduced another symbol t', which was given by the equation

$$t' = a(t - ux/c^2) \quad . \quad . \quad . \quad . \quad (3)$$

then from Maxwell's equations in E, H, x, y, z and t can be derived equations of *exactly* the same form in E′, H′, x', y', z' and t'. Thus the *mathematical* problem was solved, but there was one serious difficulty as to the relevance of the solution to the physical facts. Lorentz referred to t' as the "local" time for an observer situated in the moving frame and occupying the position x, y, z in space at the "true" time t; but he never suggested that it would in any sense be the time observed

by such an observer with any of his appliances. Any difficulty about accepting x' as a real *relative* co-ordinate of position had never arisen in view of the Lorentz-Fitzgerald hypothesis of contraction, but there was no corresponding hypothesis concerning " contraction " or " expansion " of time on a moving body to permit a rational being to accept t' as an actual measurement by appliances in motion of an interval of time which would be measured as t by precisely the same appliances at rest. It is to the genius of Einstein that we owe that amazing suggestion. But, of course, the enormous importance of his suggestion lay not in the mere enunciation of such a thesis, but in the clear and cogent reasons which he advanced in support of it. It has already been stated that the equations of optical and electromagnetic theory as formulated mathematically by the moving observer in terms of x', y', z', t' turn out to be of the same form as those formulated by the fixed observer in terms of x, y, z, t. This is highly satisfactory in so far as it is consistent with the negative results of all experiments for detecting motion through the ether or for detecting hypothetical changes due to this motion; but it leaves us in the position of worrying ourselves about a purely irrelevant concept, viz., absolute rest. Thus, if we solve the equations just written for x, y, z, t in terms of x', y', z', t' we obtain, after two or three easy steps,

$$\left.\begin{array}{r} x = a(x' + ut') \\ y = y'\,;\; z = z' \\ t = a(t' + ux'/c^2) \end{array}\right\} \quad . \quad . \quad . \quad (4)$$

But these could have been obtained from the former set by merely interchanging the accented and unaccented letters and changing the sign of u. The plain meaning of this is, that if the observer in the moving frame chooses to assert that it is he who is absolutely fixed, and that it is the other observer who is in motion through the ether; that his time is the absolute and the other " local," there is not a particle of evidence to settle his claim one way or another. Einstein in 1905 pointed out this remarkable property of the Lorentz equations (not only those written above concerning co-ordinates and time, but still further equations of his theory relating to each other various physical magnitudes as measured in the two frames), viz., that all question of velocity in a postulated absolute space had faded away; that any difference in the measurement of physical magnitudes by two observers was determined entirely by their *relative* motion to each other, and

in no way depended on their assumed absolute velocities through space. In effect, whatever importance the philosopher may give to the concept of absolute space, it is a notion of no value to the physicist; discussion of it is entirely irrelevant to his special work. Not only is this true for the concept of absolute space, it is also true for that of absolute time. "Common sense" shrinks from such a conclusion as paradoxical; yet, as we hope to show very shortly, it is quite easy to deduce it if one is prepared to grasp and accept Einstein's views on the measurement of time.

As we have seen, equations (2) and (3), or their equivalents (4), were set up in the first instance in an attempt to solve a definite problem in electromagnetic theory; yet, as Einstein pointed out, they are just the equations we shall arrive at if we abandon all considerations of an absolute space, or of any special theory of matter and electromagnetic phenomena and simply consider the reactions on pure kinematical reasoning of the underlying fact that light has the same velocity relative to all observers, whatever be their own velocities relative to each other.

A person sitting in a street car will say that he is in the same place during his journey if he is only thinking of the car, but not so if he is thinking of the street. Likewise, if he is standing in the street, he will say he remains in the same place if he is thinking of the earth, but not if he is thinking of the solar system; and there is no ascertained limit to such a sequence of statements. In fact, the phrases, "in the same place," "in different places" have no exact physical meaning except in relation to some specified frame of reference. An object remaining in the same place in one frame is changing place in another. It is one of the most notable features of Einstein's work that he has pointed out a similar lack of precision in the phrase, "at the same time," when used without consideration of a specified frame of reference. Over the earth we keep time by mechanical appliances, such as clocks, in which some periodic occurrence is definitely related to one rotation of the earth relative to the stars. However, these appliances require frequent adjustment, and in the last resort, if the order of accuracy desirable for time measurement required it, adjustment would have to be made by means of light signals, and this method would have to be adopted in any case if observers were inaccessible to one another. One observer at a pre-arranged reading of his time-piece would despatch a flash of light which would reach another observer. The latter, *after allowing for the*

time required for light to travel the distance, would set his clock accordingly. It is in the italicised words that the essence of Einstein's method of time measurement lies. For example, if the two observers be at relative rest to one another, the second observer divides their ascertained and permanent distance apart by the velocity of light, thus finds the time which the light takes and allows for that amount. If, however, the observers be in relative motion, the second observer realises that in his frame of reference the first observer (the despatcher of the signal) is moving. He must ascertain what place in his frame the sender occupied at the instant the signal was despatched, and divide the distance from the place to himself by the speed of light in order to make allowance. But what speed? According to earlier views, it would have been a modified speed obtained by combining in a well-known way the (assumed) absolute speed of light in space with his own speed in space. Were that the case, no trouble would have arisen; but, as Einstein postulated on the strength of the experimental evidence, that cannot be; he must divide by the same speed of light as before—always the one and only speed as ascertained by all observers, no matter what the assumed absolute motion may be. The effect of insisting on this can be seen from a simple example. Suppose we have two observers at rest in one frame of reference, say at some distance from each other on a long straight road. From a lamp fixed in the road and midway between them a flash of light is despatched outward by uncovering a shade for as brief a time as possible. The arrival of the light at the one observer's station will be simultaneous with the arrival at the other's—simultaneous, that is, for them—because they will say that the lamp was equidistant from them when the light was despatched, and light travels at the same speed in all directions. But now consider another frame of reference, a long lorry, for instance, moving along the road, so long that a man at one end of it would just be passing one road observer as the light reached him, and another man at the other end of the lorry would be passing the second road observer as the light reached him. For clearness, suppose the road lies east and west and the lorry is travelling east, and let us speak of the east and west roadman and the east and west lorryman. The two lorrymen would admit that as the light takes time to travel, they must have individually been a little westward of their respective roadmen when the light was despatched, since they were abreast when it arrived. That being so, the west lorryman

was then a little further away from the point of despatch than the east lorryman. In other words, the lamp was not midway between them when the light signal started. Consequently, the arrival of the light at one lorryman's place is not simultaneous with the arrival at the other, *because the velocity of light is the same eastward or westward* (or in any direction) for all people on the lorry. *On the older view*, the east lorryman receding from the lamp would have said that the light is advancing to him more slowly than to the west lorryman who is moving towards the lamp, and so the shorter distance would be compensated by a smaller velocity of light and the longer distance by a larger velocity. But in the face of the experimental evidence this hypothesis concerning variation in the velocity of light is untenable, and so the conclusion is quite valid that, whereas the roadmen agree that the light reached the momentarily common position of the east roadman and lorryman and that of the west roadman and lorryman "at the same time," the lorrymen will agree that it reached the momentarily common position of the east lorryman and roadman "at an earlier time" than that of the west lorryman and roadman.

Many will feel that the rather impracticable nature of the illustration is detrimental to the conclusion. Not so—it is merely an attempt to sketch in words the vital feature of the mathematical analysis employed by Einstein. Further, some may say that the minute times involved in the transference of light would be quite unmeasurable by any of our appliances. That is not a valid objection. The physicist's aim is to lay down principles and develop analysis thereon which will survive any practical test, however severe and precise. Whatever experimental evidence is available (and some of it is extremely searching) justifies Einstein's postulates. It is possible, perhaps, to raise philosophical objections to Einstein's method of relating time by light signals, but any physicist who accepts it as a basis for *physical* time must accept Einstein's conclusions. There is no evasion of the logic possible.

When anyone has grasped the inner meaning of this "paradox," i.e., realised that it is not a paradox, he will find that two other well-known "paradoxes" of Einstein's treatment of space and time can also be revealed to him. We will adopt the same fanciful frames as before. A flash of light is emitted from the lamp, and a little later another flash is emitted, and the interval between the flashes is adjusted so that the departure of the second signal from the lamp is simultaneous with the arrival of the first flash at a place on the road lying

to the east of the lamp. If the reader has really grasped "paradox" number one, he will at once ask, "To whom are these events simultaneous?" Let us suppose they are simultaneous to observers on the road, i.e., if a man stationed on the road at the place determines the interval between the arrival of the first and the second flash, he finds it to be equal to the time required for light to travel the distance between the lamp and his position. But now consider two men on the lorry so situated that one passes the lamp as the second flash leaves it, and the other just passes the place on the road as the first flash reaches it, the second lorryman being naturally to east of the first. From what we have just reasoned out, the arrival of the first flash at the place and the despatch of the second flash from the lamp are not simultaneous events to the lorrymen; for them the former event occurs earlier than the latter. That is, they put the despatch of the second flash from the lamp further on in time from the despatch of the first flash than the roadmen do, or they find a greater interval of time between the flashes than the roadmen. Of course, it must be noted that these two events, viz., the emissions of the two flashes occur at the same place for the roadmen, but not for the lorrymen, who see the lamp moving westward between the flashes. We can put this conclusion into formal language as follows: Two events occur at the same place in a definite frame of reference, but at different times, and the interval of time between them is measured by observers in this frame. With reference to observers in another frame moving relative to the first, these events occur at different places; moreover, the interval of time between the events as measured by the observers in the latter frame is greater than that measured by those in the former.

The third "paradox" concerns the measurement of the distance between two marks on the road by roadmen and lorrymen respectively. Call the marks fixed on the road A and B respectively, B lying to the east of A. Road observers at A and B agree that at a prearranged time on their clocks, they will make two chalk marks on the lorry as it passes them. These marked places we call C and D, and so the passage of C past A and D past B are simultaneous events—*to the roadmen*, and they will say that C to D is the same distance as A to B. What will the lorrymen say? Why, that D passed B earlier than C passed A (because, according to the first of our three propositions, events which are simultaneous to people on the road are not simultaneous to people on the lorry), i.e., that

some mark E on the lorry to the west of D passed B at the same time as C passed A. Hence they will return the distance AB as equal to CE, which is shorter than CD. So we add another "paradox" to our formal statements. Two events occur at the same time in a definite frame of reference but at different places, and the distance between the places is measured by observers in this frame. With reference to observers in another frame moving relative to the first, these events occur at different times; moreover, the distance between the places as measured by observers in the latter frame is less than that measured by those in the former.

Anyone who can grasp the standpoint from which these three propositions are developed is in a position to realise the difference between it and that of the earlier work of Lorentz and Larmor. For those who maintain the existence of the universal ether as an absolute frame of reference, a body at rest in the ether has certain linear dimensions which are the same for all observers, whether they are at rest in the ether or not. If the body is moving through the ether, its dimensions at right angles to the motion are unchanged, its dimension parallel to the motion is shortened; but this shortening is the same to all observers whether at rest in the ether or not, i.e., all observers obtain the same shortened length. The shortening is a matter dependent solely on the absolute movement of the body through the ether, and has nothing to do with the relative motion of the observer to the body. It is an absolute property of the body. For the relativist, however, motion through an ether is irrelevant. The dimensions of a body are different for different observers. For an observer at rest relative to the body, they have certain values, for an observer moving relative to the body the dimension parallel to the relative velocity is smaller, and the shortening is more marked the greater the relative velocity. In fact, it can be demonstrated by application of mathematical reasoning to the case considered above that a length which for a relatively stationary observer is l, is for an observer with a relative velocity u, $l(1 - u^2/c^2)^{\frac{1}{2}}$ where c is the velocity of light, provided the length is parallel to the relative velocity. The size of a body measured by a relatively stationary observer is greater than that measured by any other observer, and is for convenience referred to by relativists as its "proper" size; but the word has nothing of an absolute nature about it. In short, length, area, volume are all *relations* between an observed body and the person observing it.

Similar remarks apply to measurement of time. Lorentz

found it convenient to introduce "local" time for each observer; but as he looked at it this was merely a mathematical artifice in order to enable the moving observer (that is, moving absolutely in space) to summarise his facts in equations possessing the symmetrical and simple forms obtained by the observer at absolute rest, using true and absolute time. Further, an interval of local time between two events depends on the interval of absolute time between them and the velocity of the locality through the ether. On the other hand, Einstein takes "local" time as genuine physical fact and not a mere artifice; further, he holds that no observer can claim that his "local" time is absolute in contradistinction to all others. An observer at rest in a locality where two events occur will find a certain interval of time between the two events. Any other observer in motion relative to the locality will find a longer interval of time between the same two events. It is convenient to refer to the measurement of the relatively stationary observer as the "proper" interval between the events, and it can be shown that the measured interval for the relatively moving observer is obtained from the "proper" interval by division with the expression $(1 - u^2/c^2)^{\frac{1}{2}}$, i.e., the connection between the two measurements is solely a matter of relative velocity, and independent of any assumed ethereal velocity. Once more the time elapsing from one event to another is a relation between the events and the person observing them.

It should be unnecessary to remark that these conclusions are not in any way dependent on the mere adjustment of timepieces to local "origins" of time, such as the choosing of local noons on our earth. And, of course, all material standards of length and time are to be such as agree perfectly with each other when juxtaposed and *in relative rest* to each other. But it may be as well in connection with the example used above to note the fact, which rests on the purely reciprocal nature of the theory, that as the road is moving westward relative to the lorry, two events which are simultaneous to lorrymen are not simultaneous to roadmen; the event occurring at the westward place is prior to the other for the roadmen (in general, prior at the place which is in front of the other with regard to the motion of the "moving" frame relative to the conventionally "fixed" frame). Also, if the lorrymen measure a certain interval of time to have elapsed between two events which occur at a place *fixed on the lorry*, the roadmen measure a longer interval between these events, and the roadmen will

(5) or (6), so that the relations between the forces on the particle in each frame can be determined. We use $\mathbf{F}$, F_1, F_2, F_3 for the force and its components in S, and similar accented letters in S′. We also require a symbol for the "activity" of the force on the body, i.e., its rate of doing work on or of adding kinetic energy to the body. This is measured by the scalar product of $\mathbf{F}$ and $\mathbf{v}$; we shall denote it by A. The relations are then

$$\left.\begin{aligned} \beta' F_1' &= \alpha(\beta F_1 - u\beta A/c^2) \\ \beta' F_2' &= \beta F_2\,;\ \beta' F_3' = \beta F_3 \\ \beta' A'/c^2 &= \alpha(\beta A/c^2 - u\beta F_1/c^2) \end{aligned}\right\} \quad . \quad . \quad (7)$$

or, in brief, βF_1, βF_2, βF_3, $\beta A/c^2$ are also cogredient with x, y, z, t.

By means of (5), (6), and (7) it is possible to determine the relation between force and acceleration in the frame S′ if the law connecting them as measured in S is known. It appears that if the law in S is one of simple proportionality, then no such simple law will hold in S′, and *vice versa*. Such a law, therefore, does not satisfy the principle of Relativity. But after all, the proportionality of force and acceleration is not a complete statement of Newton's laws of motion. The full statement of the second law is that force is proportional to the rate at which *momentum* is changed, and as momentum is the product of mass and velocity, it is clear that the proportionality of force to acceleration is dependent on the further assumption (which Newton certainly made) that the mass of a body *regarded as a measure of its inertia* is an absolute quantity which does not depend on its condition as regards any other physical property, and in particular on its state of motion. But before the advent of the Relativity hypothesis physicists had, in developing the electrical theory of matter, been compelled to abandon the independence of mass and velocity in the case of the electron, and Einstein was able to show that a complete abandonment of it for any body was required in order to bring dynamical laws within the scope of Relativity. In short, if Newton's assumption that the mass of a particle is invariable and independent of its velocity be replaced by the assumption that in a definite frame the mass increases with the velocity in such a way that it varies as β or $(1 - v^2/c^2)^{-\frac{1}{2}}$, mathematical reasoning proves that the proportionality of force to rate of change of momentum (not, of course, to acceleration) is a law of motion which is true in any of the frames of reference, if it

be true in one of them. For a velocity which makes the ratio v/c small, the difference between the results of Einstein's law and Newton's is experimentally inappreciable, but in the case of the electron, for which v/c is not negligible, this law of variation of mass with speed is just what is required to reconcile the observed motion with the deductions of theory.

It is advisable once more to consider closely the difference between the new point of view and the old. To be sure, Lorentz's electron theory involves a variation of the mass of the electron, and the mathematical formula is identical with that of Einstein; but in Lorentz's formula v is a velocity through the ether. An electron with a mass m, when at rest *in the ether*, has a mass $m\beta$ when moving through the ether with velocity v. Its mass changes, but its values for a given state of motion in the ether is the same for all observers, no matter what are the frames from which they make their observations on the movement of the electron. On Einstein's view, the mass is m to any observer who is at rest relative to the particle ("proper" mass), and $m\beta$ to any observer to whom the relative velocity of the particle is v. Mass is a relation between the observed thing and the observer; absolute motion has no significance whatever in connection with it.

Another striking result follows from Einstein's law of motion. It is found that the work done by a force on the particle in increasing its velocity from zero to v is $m(\beta - 1)c^2$. This is, of course, the measure of the particle's kinetic energy. Hence we see that the kinetic energy of the particle is proportional to the excess of its mass for the given velocity over its proper mass. This proportionality of increase of energy to increase of mass is again quite in keeping with the most recent findings of the electrical theory of matter, which actually suggest that the concepts of energy and mass should be fused into one, that even what we call the proper mass of a body is a measure of the intrinsic energy of the complex electrical system consisting of the nuclei and electrons of its atoms. So far from opposing such a doctrine, Relativity accepts it and gives it a significance which is wider inasmuch as it postulates no special theory of the constitution of matter.

Any student of Physics will realise that if the laws of Dynamics and the laws of Electromagnetism can be expressed in forms which are valid for any one of a group of reference frames in uniform motion relative to each other, then a great advance has been made in exhibiting the conformity of all physical phenomena to the principle of Relativity. For it

the electrical theory of matter is correct in explaining such dynamical concepts as inertia and force in terms of groups of subatomic charged corpuscles (nuclei and electrons), then it follows that the movement of such particles, and therefore the properties of matter in bulk, must follow laws which are independent of the frame of reference chosen. There is one significant exception, however. Gravitation is a phenomenon which by reason of its very simplicity and universality has so far stood outside any special theory of matter. Moreover, and this is a point of greater importance at the moment, the law propounded by Newton for its measurement fails to satisfy the criterion which is required by equations (7) above. The law is, in fact, based on action at a distance, and expresses the force in terms of distances between individual particles of matter. Shortly after the appearance of Einstein's first paper, and when this anomalous position of gravitation had been recognised, Poincaré attacked the problem of modifying the Newtonian law so as to render it invariant for all the frames connected by a Lorentz transformation. He showed that a modified law which does not contradict astronomical observation is possible, and removed the old difficulty raised by Laplace that gravitation must be propagated with an infinite speed, an assertion which would, if true, place gravitation outside the Relativity principle at once. But while Poincaré's work was significant from the standpoint of the *restricted* principle of Relativity, it still left unexplained certain discrepancies between observation and Newtonian theory, e.g., the anomalous motion of the perihelion of Mercury, and, as a matter of fact, it was within a few years completely overshadowed by a further brilliant generalisation made by Einstein himself, who pointed out that the problem of relating gravitation and Relativity was intimately bound up with the problem of extending the principle of Relativity to cover any kind of relative motion (not merely uniform) of the frames of reference.

A few pages back the reader was asked to imagine the course of events in a moving laboratory, whose speed along a straight track is uniform. Let us now consider events in this laboratory if the speed of the laboratory be uniformly accelerated, say in an easterly direction. Every one who can recall his sensations in a tram or railway car, whenever it is starting or stopping, will find no difficulty in admitting the following statements. Everything in this laboratory would experience a tendency to move *through the laboratory* towards the western wall *with an accelerated speed*. That is the natural

way in which a laboratory inmate would express his experiences. Of course, outsiders would say that both the laboratory and internal objects are moving eastwards, but whereas the speed of the laboratory is increasing, that of the internal objects, unless they are attached rigidly to walls, ceiling or floor, or affected by friction with these, remains unchanged, and thus the western wall is overtaking them. The facts are the same; the mode of expression depends on the frame of reference from which one views them. In the laboratory a pendulum would not hang along the original vertical, but would rest along a direction oblique to the floor, approaching the western wall nearer at its bob than at its point of suspension, and would oscillate about this line if disturbed; unsupported bodies would fall in this direction; water in a vessel would bank up on the western side of the vessel so as to preserve a plane surface, but at right angles to this same direction. Everything in the laboratory would be affected in a similar way. If friction were sufficiently small, loose bodies on the floor and benches could only be kept from moving with increasing speed towards the western wall by fastening them to the eastern wall or by some similar device. In fact, it would not be unnatural for the laboratory population to say that for some reason or other gravitation was acting towards the western wall as well as towards the floor. If no vision outside their laboratory were possible they might even imagine that they were actually at rest on the earth as before, and that some influence had pushed the laboratory up at its eastern side, and so from east to west was "downhill." If, however, they investigated the period of oscillation of a pendulum, they would find it to be shorter than before, and thus their "force of gravitation" would be not only altered in direction *to them*, but would also be stronger. Of course, an outsider would refer to the new "horizontal component of gravitation" as "fictitious." That would be quite right from his point of view. But from the laboratory point of view, the new component would be quite as real as the original, for, in the last resort, gravitation is a name introduced for the purpose of conveying to the mind a certain common feature of a wide range of mechanical phenomena. It does not purport to explain the "why" of those phenomena, despite a rather popular, but fortunately fast waning, belief to the contrary. To the person in the laboratory everything tends to move towards the western wall with a definite acceleration of speed in that direction, and that is the unique feature of a "gravitational field" wherever it is encountered. The imme-

diate point is that he can still summarise all his mechanical experiences in the laws of motion, provided he uses as his force of gravitation, not the force familiar to the person at rest on the earth's surface, but the force as modified by the new westerly component. Moreover, if another laboratory passed by his own, and he could see into it and make observations on events there, he would probably perceive that the gravitational force in laboratory number two is different to that in his own, both in magnitude and direction; but this would give him no clue as to whether his laboratory is at rest and the other is absolutely accelerated with regard to his, or *vice versa*.

It may occur to the reader that if gravitation can be modified in this way by merely transferring our thoughts to another frame of reference, it may be possible to "modify it away altogether." It certainly is, though if we transferred ourselves and not our thoughts to some of the frames the results would probably be fatal. Recall the familiar "sinking sensation" when one is in a lift and it is just beginning its descent to a lower level. During a large part of the journey, the sensation is absent, because the lift is descending uniformly; but for a brief interval at the start the lift is picking up speed, it is being accelerated downwards, and during this interval the pressure of everything on the floor decreases. In fact, if the gear broke and the lift fell freely down the shaft with the natural acceleration of gravity, nothing on the floor would press on it; if any unfortunate person in the lift ceased to hold any article in his hand, it would not fall *to the floor.* No doubt other people would say it was falling, and just in the same manner as the lift; but they are referring experience to another frame. People in the lift would for a short time at all events experience none of the familiar effects of gravitation. Nothing would fall *in the lift;* articles in motion would move in straight lines *through the lift;* the floor would cease to press people upwards—the most persistent and universal appeal of gravitation to our senses—and very uncomfortable they would be when missing this familiar pressure, even apart from the dread of the final crash. We can relate this to the moving laboratory, where an eastward acceleration of the laboratory relative to the earth introduced into the laboratory a westward "force of gravitation" which did not exist in the outside frame. So here, a downward acceleration of the lift relative to the earth introduces into the lift an upward "force of gravitation" (non-existent in our usual frames), and this just compensates the already existing downward force and so results in the absence of

gravitation, i.e., absence of the familiar features of falling bodies, curved trajectories of projectiles, supported bodies, etc. It is of interest to note in passing that quite the opposite effect takes place at the conclusion of a lift descent, i.e., a usual descent and not the disastrous episode pictured above. Then the lift is slowing up, and this is equivalent to an acceleration upwards relative to the earth, introducing a downward " force " into the lift in addition to the usual one, and resulting in a brief increase of pressure of ourselves on the floor, and an increased acceleration in the speed of any unsupported body towards the floor.

Let us abstract ourselves from the fatal surroundings of the dropping lift ; let us put the lift—or, for convenience, a more commodious room like our original laboratory—out " in space." It will be admitted that if anywhere near the solar system, the laboratory would be moving relatively to the parts of this system. It might be " falling towards the sun," or perhaps towards one of the other members, if near enough ; on the other hand, we could avert the still impending, but postponed, disastrous collision by arranging to have it moving in an orbit, just like a planet, satellite, asteroid, or comet. We will waive the difficulty as regards the subsistence of life in such peculiar circumstances. M. Verne and Mr. Wells have overcome little troubles like that for our amusement and delight. Having, like those intrepid *voyageurs*, Servadac and Cavor, " changed our frame of reference " not merely to a comet or the moon, but to something still more exiguous, an " asteroidal laboratory," whose own gravitational influence is too feeble to be considered, we can at leisure and without unpleasant anticipations study the conditions in our new abode. After a time, we would become accustomed to the " sinking sensation," i.e., the absence of pressure on the floor ; indeed, there would be no floor. Gravitation would be absent in our room ; we would have to take great care that an incautious muscular effort when in contact with any of the walls did not project us violently against the opposite wall or result in permanent banishment from our new home, if an opening in the wall allowed us to pass through to the unpleasant fate of becoming a lonely asteroid. In short, our frame of reference would be " gravitation-free." We have not necessarily to carry our thoughts to enormous distances from suns and stars to form the idea of a " place where gravitation does not act."

We thus can conceive of the existence of frames of reference

in which over a limited region gravitation effects are practically absent in the sense that accelerated motion *relative to this frame* is not a natural and striking feature of moving bodies as it is relative to earth, for instance. If in such a frame we should desire to reproduce the familiar features of gravitation, we could do so by moving a second frame through it with accelerated velocity. On an asteroid, for example, " natural " gravitation would be almost absent, but if our laboratory were running carriage-wise along its surface with an accelerated speed, we would produce within the laboratory a " gravitational field " in a direction opposite to the laboratory's acceleration, this being the only gravitational field in the laboratory now, and not merely an additional amount as it was when we conceived the operation carried out on the earth. The conclusion to which this line of thought leads is the relativity of gravitation in so far as mechanical effects are concerned. If we have not merely to admit inability to measure a velocity in absolute space, but also inability to measure absolute acceleration, we cannot say how much of our actual gravitational field is " real " and how much is " fictitious " and due to such an acceleration. Indeed, in the absence of such knowledge, the idea of an absolute acceleration of the earth in space is as irrelevant for physical science as the idea of an absolute velocity. There is, however, one qualification to be referred to at this point. We do measure a rotation of the earth on its axis *with reference to the so-called fixed stars.* Moreover, we also measure a rotation by means of such apparatus as the Foucault pendulum, where we do not postulate the fixed stars as a frame of reference ; in fact, this rotation is the rotation of the plane of vibration of the pendulum relative to the objects in the laboratory ; but our minds being prepossessed with the idea of an absolute space, it has been the custom to refer to the Foucault effect as due to the rotation of the earth in an absolute space. It so happens that the two rotations agree experimentally, which has led to an assumption that we can use the mean of the " fixed stars " as a frame of reference which may be treated as absolute in so far as rotation is concerned. On such an assumption it is clear that our gravitational field would contain a small " fictitious " part, due to centrifugal action, which would be at right angles to the axis of rotation and equal to $\omega^2 r$ where ω is the angular velocity of the earth and r the distance of the part of the earth considered from the axis. The remaining part would, on such a view, be the " real " gravitational field (unless there be some other " fictitious " part due to an acceleration of the translatory

motion of the earth in absolute space). Needless to remark, such separation of a gravitational field into real and fictitious parts is quite foreign to the standpoint of Relativity. The field depends on the frame of reference, and the change involved in passing from one frame of reference to another must be calculated, not in terms of unobservable movements in an absolute space, but in terms of observable relative velocities or accelerations. Thus the two fields referred to above would differ from each other by a term depending on the relative angular velocity of a frame not turning with respect to the fixed stars and a frame, like the earth, which is so turning. It is true that the mathematical form for this term has been arrived at by reasoning which seems to postulate an absolute rotation in advance. But after all, we are not committed to the absolute validity of this form. It is, indeed, like many other results deduced from the Newtonian dynamics, a close approximation in the actual cases where we make practical use of it, but if extrapolated beyond these limits it lands relativist and non-relativist alike in the difficulties of conceptually infinite space; and, although this is a point which cannot be treated here, but must be deferred until the end of this book, the relativist does not make such an extrapolation, but has indicated more closely the mathematical method which must be pursued in order to obtain the correct relation between the field in one frame and the field in a second, having a definite angular velocity relative to the first.

Mechanical effects apparently give no decided evidence on the existence of an absolute acceleration any more than of an absolute velocity. It is true also that optical and electromagnetic experiments yield no support to the existence of an absolute velocity. But there still remains open the possibility that optical experiments might provide evidence in favour of the view that absolute acceleration is an observable phenomenon. For instance, a ray of light as observed by two people each using a frame of reference accelerated with respect to the other, would have two different geometric forms whose relation to one another could be calculated in terms of the known relative acceleration. Now, if a ray of light be observed in one gravitational field, and another ray in another field differing from the former by an amount equivalent to the relative acceleration just mentioned, it is clearly a matter of experiment to determine whether the two rays have the same geometric form or not, and if not, whether the relation between the forms is the same as in the previous case or not. If the forms are

the same or if they differ in a manner which is not equivalent to the differences introduced by simple change of reference frame, then although "real" and "fictitious" gravitational fields would be equivalent in their effects on mechanical phenomena, they would not be equivalent in their effects on optical phenomena. What Einstein postulates is that they are completely equivalent, and so adds the "Principle of Equivalence" to his previous postulates.

It follows from this that a ray of light should be deviated in passing near a large gravitating mass, just as a particle of matter would be if it had the same speed. But the reader must be careful at this point. He has just learnt that the behaviour of a particle of matter as deduced from Relativity dynamics would be very different from that deduced from classical dynamics. On the older theory, combined with the Newtonian law of gravitation, the principle of Equivalence would yield a deviation of a little less than a second of an arc in the case of a ray of light passing close to the sun. But Relativity dynamics combined with Einstein's law of gravitation (to which reference will be made presently) gives the deduced deviation as just double the former amount. So observation is not so much concerned with verifying the fact of a deviation, which is quite compatible with older views, but in obtaining as exact a measurement as is possible. The eclipse expeditions of 1919 certainly gave considerable support to the Einstein value of nearly two seconds, but the matter is so important that further verification is eagerly awaited.

To grasp the full import of Einstein's law of gravitation and comprehend the attitude of mind which rejects the Newtonian law as inadequate, and accepts Einstein's as a more complete summary of our knowledge, is impossible without the aid of mathematical analysis. Unfortunately for semi-popular explanation, the sort of analysis required is unfamiliar even to those who have taken the customary University courses of instruction. It is not that it is especially difficult or full of artifices and "dodges"—far from it. It is plain and straightforward enough, but in a word, it has not usually been taught in the past, whatever may happen in the future. At this stage we can merely state that Einstein's theory is a hypothesis concerning the "curvature of space-time," and proceed to adumbrate what that peculiar phrase means.

We are familiar with the treatment of geometrical figures on a plane surface by means of Cartesian or polar co-ordinates. Certain readers may also be aware that there is an analogous

treatment of figures on a spherical surface. As a matter of fact, there exists a very important branch of geometry which treats of figures on any surface in terms of co-ordinates which were first introduced by Gauss. Suppose we have a certain surface S, and that we consider the curves in which it is cut by a surface

$$f(x, y, z) = u$$

where u is a parameter and x, y, z are ordinary three-dimensional Cartesian co-ordinates. If we vary the value of the parameter u we obtain a family of curves on S. Consider another family of surfaces,

$$\phi(x, y, z) = v$$

where v is another parameter; this gives another family of curves on S. If we assign definite values to u and v, we define a point on the surface S, and we can call u and v Gaussian co-ordinates of that point; the aggregate of the curves formed by the intersection of S and the f surfaces, and of S and the ϕ surfaces, constitutes a mesh on S. Of course, we can "transform the co-ordinates" by changing the mesh system, i.e., by altering the functional forms of f and ϕ; so that assigned numerical values of u and v would not define the same point on S as before.

Let us consider two points, P and Q, on S, which are very near one another, one being defined on some mesh system by definite values of u and v, and the other by $u + \delta u$, $v + \delta v$. It can be shown that the square of the element of length PQ is equal to

$$g_{11}\delta u^2 + g_{22}\delta v^2 + 2g_{12}\delta u\delta v,$$

where g_{11}, g_{22}, g_{12} are three functions of u and v, whose functional forms will depend on the form of the surface S and on the functional forms of f and ϕ. The length of a curve drawn on the surface between two finitely separated points A and B is given by

$$\int_A^B ds = \int_A^B (g_{11}du^2 + g_{22}dv^2 + 2g_{12}dudv)^{\frac{1}{2}}$$

the line of integration being the curve itself. Two very important results can now be quoted. First, defining a "geodesic" as a curve on the surface such that the length of it between any two points A and B is either greater or less than that of all neighbouring curves drawn on the surface between A and

B, we find that the equations of a geodesic are two differential equations in d^2u/ds^2, d^2v/ds^2, du/ds, dv/ds, in which the coefficients are definite mathematical functions of the g_{11}, g_{22}, g_{12}, and their differential coefficients with respect to u and v. We may alter our mesh system, i.e., alter the functional forms of f and ϕ in x, y, z, and thus transform our co-ordinates and alter the functional forms of g_{11}, g_{22}, g_{12} in u and v, yet the differential equations of a geodesic retain the same mathematical form; they are "covariant." The second result concerns the "curvature" of S, which is defined as follows. Consider a small element of area surrounding a point P on S, and draw normals to S at all the points on the boundary of this area. From any arbitrary point draw a series of lines parallel to these normals, forming a cone which will cut out an area on the surface of a sphere described round the arbitrary point as centre. If the radius of the sphere is so adjusted that this area is equal to the original element of area on S, then S is said to have *at the point* P the same curvature as this sphere, and the quantitative measure of this curvature is taken to be $1/R^2$, where R is the radius of the sphere in question. Gauss succeeded in showing that this curvature is equal to a certain mathematical expression involving the g_{11}, g_{22}, g_{12}, and their first and second differential coefficients with respect to u and v; and, once more, the expression is independent of the functional forms of f and ϕ in x, y, z, i.e., of the mesh-system or system of co-ordinates. He was further able to show that any deformation of S *which does not involve straining, crumpling or tearing* does not alter the values of the curvature at each point, so that, for example, the curvature is zero not only for plane surfaces but for all surfaces which can be developed into a plane; and also that if a surface have finite values of curvature everywhere, it cannot be developed into a plane.

Some years later Riemann put forward the idea that our own space might require similar general treatment; that it is not a foregone conclusion that physical measures of distance, if sufficiently refined, would agree with the conclusions deduced by geometrical treatment based on Euclidean assumptions, and, in particular, on the parallel postulate. It might, he argued, be necessary for a more complete treatment of geometry to postulate that the distance between two neighbouring points is given by *a general quadratic function* of the three differences of the co-ordinates involving as coefficients six functions of the co-ordinates, and not by the simple quadratic form employed in the customary treatment, viz., the sum of the squares of

the co-ordinate differences. It may be as well to warn the reader at this point that, of course, if we adopt polar co-ordinates, ellipsoidal, cylindrical, or any other variety of curvilinear co-ordinates, we do have to employ a general quadratic function of co-ordinate differences; but there is always implicit in conventional geometrical reasoning the hypothesis that it is possible to choose a system of co-ordinates, the Cartesian, for which the expression takes the form with constant coefficients. Now it is just this hypothesis which Riemann questions. We certainly cannot do the analogous thing in two dimensions on a surface with essential curvature. We can only be sure by direct physical measurement of sufficient precision that we can or cannot do it in any three-dimensional geometry which has to be employed in reasoning about physical phenomena. The main conclusion as regards this part of Riemann's work is this. If we consider the mathematical expressions which are the natural generalisation for *three* co-ordinates of Gauss' single curvature expression for two, and are, like Gauss' curvature, expressions involving the six g-coefficients now required and their first and second differential coefficients with respect to the co-ordinates, then if these expressions are zero, space is Euclidean, flat, "homaloidal;" if they are not zero, there is an essential "curvature" in space, which may be negligible over a small enough region (just as a small portion of a surface may be regarded as nearly flat), but cannot be neglected over a large region without some contradiction with physical fact arising.

Einstein's treatment of the general relativity of space, time, gravitation, and electromagnetism depends in general on the adoption of the same attitude towards the four-dimensional continuum formed by physical events, and in particular on the employment of the same mathematical method, adapted to deal with four co-ordinates instead of two or three. In the previous pages we have seen how the spatial co-ordinates and the time co-ordinate of any physical event are more closely connected than has been suspected hitherto and how the transformation of co-ordinates involved in changing one's frame of reference, in general, affects all four. The aggregate of all physical events we call "space-time," or the "world." Now, two events in this world have a certain "interval" or "separation" between them which is capable of mathematical definition. From an examination of equations (2), (3), or (4), will be found that

$$c^2\delta t^2 - \delta x^2 - \delta y^2 - \delta z^2 = c^2\delta t'^2 - \delta x'^2 - \delta y'^2 - \delta z'^2$$

where x, y, z, t and $x + \delta x, y + \delta y, z + \delta z, t + \delta t$ refer to two defined events in one frame of reference, and x', y', z', t' and $x' + \delta x'$, etc., refer to the same two events in another frame. It appears, therefore, that two definite physical events possess the property that there is a definite relation between them which is determined by the special quadratic expression

$$c^2\delta t^2 - \delta x^2 - \delta y^2 - \delta z^2$$

for this expression maintains a definite value, no matter what frame of reference the events are viewed from, provided we accept the limitation of uniform motion imposed by the restricted principle of Relativity. In order to free himself from this restriction, Einstein postulates that this separation is a physical quantity independent of any frame of reference, and that its complete expression depends on a *general* quadratic function of the four co-ordinate differences. Such an expression will involve *ten* terms, and so ten coefficients, $g_{11}, g_{22}, g_{33}, g_{44}, g_{12} \ldots g_{34}$, will appear all functions of the co-ordinates in general. The actual functional forms of these coefficients will vary with a change of reference-frame, i.e., with a transformation of co-ordinates, and Einstein assumes a close correlation between the changes in these functional forms and the change in the gravitational field which is involved, and which has been previously pointed out, a correlation which can only be stated precisely in terms of mathematical symbols. Einstein succeeds in throwing the laws of kinematics, dynamics, and electromagnetic theory into equations which involve the symbols for the usual physical concepts of velocity, force, mass, energy, electric and magnetic intensity, etc., and the ten g-coefficients, and these equations *preserve the same mathematical form* no matter what change is made in the reference frame, i.e., no matter how the co-ordinates are transformed. Thus is the general principle of Relativity satisfied.

Gravitation has been omitted in the catalogue of terms just written. This has been done advisedly, for in Einstein's theory *gravitational force* has no place as a *fundamental concept*. In Newton's theory rectilinear motion in an absolute space—inertial motion—is postulated as a natural phenomenon, and so gravitational force has to be postulated also as an explanation why this natural motion is not in general adhered to. For Einstein there is no absolute space; but there are "natural" paths for a material particle in any definite frame, such paths as, in fact, it would pursue provided it be freed from air resistance, mechanical shocks, etc. They are not in general

rectilinear, but, provided the conditions above are obeyed, they are adhered to, and so no influence has to be postulated to account for a deviation from them, since there is no such deviation. If one asks what is Einstein's criterion for the " naturalness " of a path, the request can only be answered in mathematical terms. It was mentioned above that on a surface two covariant differential equations constitute the equations of all geodesic lines, and such lines do possess a unique property which singles them out from all possible lines on the surface. Einstein employs the four differential equations which are the generalisation for four variables of the equations of surface geodesics for two, as the equations of his natural paths. Of course, these equations single out, in the first instance, lines in the four-dimensional world of space and time, " world lines " of particles ; the " projection " of these on any given observer's space will be the natural spatial paths in his frame of reference.

Lastly, what of the law of gravitation itself ? Again, the reply comes that it is a set of differential equations, and the expressions involved in these equations are twenty functions which are the natural mathematical generalisation for four variables of the expressions for the curvature of a surface in two variables, or the Riemann " curvature components " of space in three. If these twenty functions were zero, Einstein assumes that either gravitation would not occur in our frame of reference, or, at all events, it could be *entirely* removed by a change of frame. But we cannot remove it in such a wholesale manner ; we can remove it, as we have seen, over a small region, just as we can regard a small element of a surface as flat. Hence there is some essential " curvature " in space-time which cannot be entirely removed by any choice of reference frame, i.e., by any change of co-ordinates. The twenty " curvature " components are not individually zero. Yet there must be some mathematical relation between them to correspond to those physical properties which are common to our space and time, no matter what frame of reference is used. Einstein made a brilliant guess that although the individual curvature components are not zero, certain linear functions of them are zero, for regions outside matter, and are equal to a similar group of functions built on the energy, stress, and momentum of matter within matter itself.

But we have reached the limit where description in general terms ceases to be of service, and it is time to take up the mathematical development of Relativity in earnest.

PART I.

RESTRICTED RELATIVITY.

CHAPTER I.

We must now begin a careful study of the conclusions which can be drawn from the definite pronouncement of experimental research, that however useful the ether may have proved to be as a conceptual medium for the propagation of radiation, it has failed to " materialise " as a system of reference privileged to assume the title " fixed," in contrast to all bodies in the universe.

It was Einstein who, in 1905, grasped the full import of this failure and realised that all physical laws must be framed so as to summarise the results of measurement and observation in the same terms, no matter what the system of reference might be, in which the observers and their apparatus are situated. At the outset of his researches, however, he confined himself to the question of framing laws in the form of equations, which, if valid for a system of reference S, would also be valid for any other system of reference moving relative to S with a *uniform* velocity. It is sometimes stated that Einstein has " abolished " or " abandoned " the ether. This is one of those dangerous half-truths which are apt to create serious misconception in the minds of those who are interested in scientific research, but are precluded for one reason or another from reading the original literature.

What Einstein pointed out in 1905 was the fact that if one adopts the Lorentz equations for transforming from axes fixed in the ether to axes fixed in a body moving uniformly through the ether, it becomes impossible to decide which of the two systems of axes is fixed in any absolute sense; indeed, it is impossible to say if either of them is fixed. This arises from the reciprocity to which attention has already been drawn.

Thus, in considering the formulation of general physical laws as differential equations valid for any system of reference, the question of the existence of the ether is irrelevant It is of no more service for that specific purpose than any body in motion with respect to it. But that does not amount to a denial of its existence. Certainly, as will be seen presently, it

becomes impossible to reconcile the existence of an immovable, "stagnant" ether with the standpoint of Relativity.

It is the demand for a formulation of laws valid for any system of axes which constitutes the starting-point of the new theory. Its first postulate reads: "The laws of natural phenomena are the same whether the phenomena are referred to the framework of reference S to any other framework S′ moving relatively to the former with a uniform velocity and without rotation." Or still more precisely we may phrase it thus: "The equations which summarise the sequences of natural phenomena have the same form when referred to a given set of axes as when referred to a second set moving relatively to the former with a uniform velocity and without rotation."

It is most important to observe that Einstein takes as his fundamental postulate a statement which appears as a final (and to some extent an approximate) deduction of the work of Larmor and Lorentz. In so doing he has introduced, as will appear in due course, a notable simplification into the analysis of dynamical and electromagnetic problems. Even before the appearance of his gravitational theory he had shown how a number of phenomena, previously explicable by the introduction of certain *ad hoc* assumptions into electromagnetic theory, could be easily deduced from the restricted principle.

In addition to the postulate enunciated above, Einstein assumes that *in vacuo* light is propagated with a velocity which is determinate for all observers, and is independent of any motion which the source may have, *relative to any given observer.*

This second postulate, while quite consistent with experimental results, cannot be readily reconciled with the view that the medium of transmission of light is a fixed and stagnant medium. It is true that the latter theory assumes that the velocity of light is independent of the motion of the source *relative to the ether;* but it also assumes, on the basis of classical relativity, that it is not independent of the motion of the observer relative to the ether. As a consequence, the "velocity of light" would refer to the velocity relative to an observer fixed in the ether, while the velocity relative to any other observer would be obtained by adding to this velocity the reversed velocity of the observer through the ether. But it is just this deduction from the classical theory which is contradicted by experimental research, unless one postulates the Lorentz contraction for matter in motion through the ether. Einstein, on the other hand, makes direct use of the experimental results, and the second postulate (the principle of

constant light velocity) which he bases on them determines the relations which must hold between the spatial and temporal measurements of different observers.

At the time at which Einstein began his researches, the evidence upon which he based this postulate was the negative result of the Michelson-Morley experiment, the principle of which was outlined in the previous chapter, a full account being obtainable in standard text-books of Optics. But the conclusions derived by combining the principle of Relativity with the assumption that the velocity of light is independent of the motion of its source are so extraordinary, that attempts were made to avoid these conclusions by substituting for Einstein's second postulate the hypothesis that the velocity of propagation is the vector sum of the real velocity of light and the velocity of the source, a return, in fact, to something akin to an emission or ballistic theory of light. Such a hypothesis would lead to a very simple Relativity theory involving no modification of our ideas of space and time, and would yield an extremely simple explanation of the result of the Michelson-Morley experiment, but being radically in contrast with the electromagnetic theory of light, obtained little favour. Still, as the existence of a medium for an undulatory propagation of light is not vital to the Relativity theory, relativists cannot offer such mere lack of popularity as a valid objection to an emission or quasi-emission theory of light. So it is necessary to refer, even if only briefly, to various suggestions which have been made in this direction, and the bearing of experimental evidence on them.

Supposing we adopt an undulatory theory coupled with the hypothesis that the velocity of light is affected by the velocity of the source in a vectorially additive manner, it can be shown that the wave-length of the light is unaffected, while the frequency of the light is altered in a manner according with the usual Döppler principle. Of course, without this hypothesis both wave-length and frequency suffer a Döppler change. Now, as the dispersion of a prism is a matter which depends on *frequency* (at least, according to all the theories of dispersion hitherto proposed), it would be fruitless to attempt to decide between the rival positions by observing the displacement of spectral lines by means of prisms. But observations with a grating depend on *wave-length*, and so if the suggested additive hypothesis were true, a grating should give no appreciable result in the study of the Döppler effect, and this is not in agreement with experience. Experiments such as those of

Galitzin and Wilip * on the stars and the limb of the sun, and of Stark † on the canal rays, agree with the conclusion that the velocity of light is constant and independent of the velocity of its source. There are two criticisms, however, which may be directed against these results. One is that astronomical observations of the Döppler effect are not always made with an *a priori* knowledge of the relative velocities of source and observer, and in observing the canal rays the measures are of small precision. The second is that in so far as mirrors which are in motion are employed, the conditions are further complicated by the fact that these cannot be compared to moving sources, and may produce different consequences. In reply to the former criticism, it has been pointed out by Comstock ‡ and de Sitter § that apart from the Döppler effect there is evidence from the behaviour of binary stars against any form of theory requiring the additive velocity, for it would necessitate a difference between the observed time of one half-rotation of each member round the other, and the time of the other half-rotation, amounting to $4vl/c^2$ where v is the velocity of rotation, c the velocity of light from a fixed source, and l the distance from the earth. Such a result has not been observed, although it appears that its detection would be quite possible if it existed. As regards the criticism of the use of moving mirrors, *emission* theories differ in their assumptions as to the behaviour of light after reflection from a moving mirror. Three assumptions have been employed in various researches. The first is that the mirror acts as a new source, and the reflected light has the same velocity relative to the mirror as the incident light had relative to its source. This has been considered by Tolman ‖ and shown to be incompatible with experiments on the velocity of light from the two limbs of the sun and with measurements of the Stark effect in canal rays. The second assumption, made by Stewart,** viz., that the velocity added to the reflected light is that of the image of the original source (so that the light has a velocity c relative to this image), has also been dealt with by Tolman, and shown to be in contradiction to the Stark effect measurements in canal rays. The third assumption has

* "Communications Acc. Russe" (1907), p. 213.
† "Ann. Physik," **28** (1909), p. 974.
‡ "Phys. Rev.," **30** (1910), p. 291.
§ "Phys. Zeit." **14** (1913), pp. 429, 1267.
‖ "Phys. Rev." **31** (1910), p. 26; **35** (1912), p. 136.
** Ibid., **32** (1911), p. 418.

been introduced by Ritz * into a rather complete emission theory. It is that the light retains throughout its whole path the component of velocity which it obtained from its original moving source, and hence, after reflection, the wave front travels out with velocity c from a point which has the same velocity as the original source had at the moment of emission. Recently Majorana † has undertaken a direct experiment on this question of the velocity of light from moving mirrors, and finds that his results authorise the conclusion that reflection of light by a moving metallic mirror does not alter the velocity of propagation of the light itself, and so disposes of the assumptions (1) and (2) above. But, as Majorana is careful to point out, only direct experiments with interferential arrangements on the velocity of propagation of light from a source set in motion artificially have any bearing on Ritz's theory, and in a later paper ‡ he describes the results of some experiments in which the difficulties against endowing a source with a large enough velocity (several hundreds of metres per sec.) and maintaining the light sufficiently monochromatic are overcome. Concerning these and similar results, he says: "From the researches made by Michelson, Fabry, and Buisson, and by myself, it results that the velocity of light is not influenced by reflection on mirrors or reflecting surfaces; from those now described by me, it results that the said velocity does not change by the movement of the source. These facts are surely in harmony with the theory of Relativity; but really, in spite of their evident interest, they cannot logically be cited as sure experimental proof of this theory. In fact, two experimental circumstances must not be forgotten: first, the presence of materials which are traversed by the interfering rays (air, glass, metals); and second, the gravitation field of our earth. While it is possible to imagine experiments entirely apart from the former, it cannot be foreseen if later experimental results will bring into evidence the eventual influence of the second."

The weight of all the available experimental evidence seems to favour the validity of Einstein's second postulate, and we shall proceed to show how the Lorentz-Einstein equations may be derived from it.

Let us consider two material bodies in uniform relative motion to one another, each body serving as a framework of

* "Ann. d. Chim. et Phys.," **13** (1908), p. 145; "Arch. d. Genève," **26** (1908), p. 232; "Scientia," vol. 5 (1909); "Collected Works," p. 317.

† "Phil. Mag.," **35** (1918), p. 163; "Phys. Rev.," **11** (1918), p. 411.

‡ "Phil. Mag.," **37** (1919), p. 145.

reference for a group of observers situated on it. Each group chooses a set of rectangular axes. Let us denote those for the first group by OX, OY, OZ, and those for the second group by O'X', O'Y', O'Z'. The co-ordinates of a point in space referred to the first set of axes we denote by (x, y, z), and if referred to the second set, by (x', y', z'). These co-ordinates are measured by standards which are assumed to agree perfectly when placed together for comparison. To observe the time of an event, each member of a group makes use of a timepiece which is assumed to be in perfect agreement with all other clocks employed by that group. All clocks *in one group* are synchronous if the time at which a light signal is despatched by an observer A (as read on his clock), and the time of its arrival at an observer B (as read on his clock), differ by l/c where l is the length of AB and c is the velocity of light, this being true for all possible lengths and directions of AB in this group. This method of synchronising the clocks in one group with each other implies that, if a light signal is despatched from a point A at time t_1 (read on a clock at A) arrives at B at a time t_2 (read on a clock at B), and is reflected by a mirror at B back to A, returning there at a time t_3 (read on the clock A), then $t_2 = \frac{1}{2}(t_1 + t_3)$. A good deal of stress is laid on this process of synchronism in Einstein's original paper, and a critical discussion of it is contained in Chapter IV. of Dr. Silberstein's book. One essential feature of it, which must be borne well in mind, is the employment by both groups of observers of the same value for the velocity of light. For by the second postulate c is a universal constant (in cm. sec. units, approximately 3×10^{10}).

By these ideal devices the observers of any given group who are supposed to be in relative rest in their framework can assign four co-ordinates to an event, viz., the three Cartesian co-ordinates of the place (idealised as a point) where the event happens, and the time of the event (idealised as instantaneous) read by an observer at the place on the clock which he possesses, or read on a given clock at the origin on receipt of a light signal from the observer, who despatches it at the instant the event happens. Of course, in the latter case the reading of the clock at the origin must be reduced by the interval required for the light to travel from the place to the origin.

It is only by the employment of such devices that we can give any definite meaning to the word "simultaneity" as applied to events in different localities. When two events are actually perceived by one person, he relies on his own judgment

and skill as an observer to determine whether they are simultaneous or not. Personal judgment in such matters is not perfect; each observer has his "personal equation." But such judgment is in the last resort independent of clocks and measuring appliances in general. The matter is quite different for events in different places which do not come under direct observation by one observer. Two observers, at least, are required. Each one must observe a simultaneity (in the personal sense) between the event in his neighbourhood and a particular reading on his clock. But until the observers have communicated their readings to each other they are not in a position to make any statement about the simultaneity of the two events, or their order in time if they are not simultaneous. If this fairly obvious consideration be borne in mind, there is no need to fear the so-called paradoxical conclusions of relativity concerning time-order.

We must now determine the relations which hold between the space and time co-ordinates (x, y, z, t) of a given event as determined by the first group of observers and the co-ordinates (x', y', z', t') of the same event, as determined by the second group.

It will simplify the analysis a little if we arrange the axes OX and O′X′ to be in one line, the relative motion of the two frame-works S and S′ being parallel to this line. It will also be convenient to assume that each group of observers chooses the original instant for time measurement to be the instant when O and O′ coincide. Let the S observers measure the velocity of the S′ framework relative to themselves as u, i.e., at time $= t$ in the S framework $OO' = ut$, O′ being, of course, on the axis OX.

Suppose that at the instant when O and O′ were coincident a flash of light is emitted from a source at this (momentarily) common origin. At a later time, the part of the wave travelling along the common axis OX will have reached an observer A of the S group situated at the point $(x, 0, 0)$ and an adjacent observer A′ of the S′ group at the point $(x', 0, 0)$. Let the reading of the A clock be t and of the A′ clock be t'.

Fig. 1 represents the state of affairs as it appears to the S group when their clocks indicate t.

In the interval t, O′ will have moved from coincidence with O into coincidence with B where $OB = ut$. Consequently, the observers in S will estimate the distance from O′ to A′ as $x - ut$, because that is the distance between two of their own observers who were respectively coincident with O′ and A′ at

the same instant, i.e. (as insisted on above), when the clocks at B and A recorded identical readings. This is the only method available for the S observers to estimate this distance; for O′ and A′ are moving past them. Of course, to the S′ observers the length of O′A′ is x' Now the relation between the lengths x' and $x - ut$ is not one of equality. Such an assumption puts us back once more into all the difficulties of classical relativity which Einstein's relativity was designed to overcome. But the relation is one which must depend, according to Einstein,

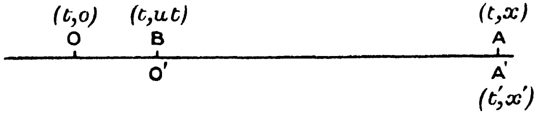

FIG I.

on the *relative* velocity of the two frameworks, and not in the least on any supposed absolute velocities in space or through the ether of the two frameworks. Owing to the simplicity of the physical relation (uniformity of the relative motion), one naturally expects the mathematical relation also to be simple. So we shall assume a simple linear relation and write

$$x' = a(x - ut) \quad . \quad . \quad . \quad . \quad (1)$$

where a is a constant to be determined.

But since light travels from O′ to A′ in the time interval t',

FIG. 2.

therefore $x' = ct'$; for all observers use the same value for the light velocity

Similarly $x = ct$, and therefore $ut = ux/c$.

Hence equation (1) can be written

$$ct' = a(ct - ux/c)$$

or

$$t' = a(t - ux/c^2) \quad . \quad . \quad . \quad (2)$$

Now let us consider the state of affairs as it appears to the S′ group of observers.

This is represented in Fig. 2.*

* The reader is warned that this is not the same arrangement as in Fig. 1. No S observer opposite O′ is put in the sketch, since none

An observer B′ of the S′ group is opposite O when the former's clock registers t', so that O is momentarily located at the point $(u't', O, O)$ in the S′ framework, where u' is the velocity of the S framework, as measured by the observers in the S′ framework.

Reasoning just as before, we arrive at the equations

$$x = a'(x' - u't') \quad . \quad . \quad . \quad (3)$$

$$t = a'(t' - u'x'/c^2) \quad . \quad . \quad . \quad (4)$$

where a' is a constant.

Equations (1) and (2) are to be consistent with (3) and (4). Hence, by inserting in (3) the values for x' and t' from (1) and (2), we obtain

$$\begin{aligned} x &= a'\{a(x - ut) - u'a(t - ux/c^2)\} \\ &= aa'\{x(1 + uu'/c^2) - (u + u')t\}. \end{aligned}$$

If this is to be true for all values of t and x, it follows that

$$u' = -u$$

and $$aa' = 1/(1 + uu'/c^2) = 1/(1 - u^2/c^2).$$

This means that each group estimates the relative velocity of the other group to have the same magnitude (but, of course, opposite directions). It follows as a justifiable assumption that $a = a'$, for each of these constants represents the ratio of the length of a rod as measured by observers at rest with regard to it, to the length as measured by observers moving parallel to it with a definite relative velocity which the latter estimate to be u in magnitude.

Hence it follows that

$$a = a' = 1/(1 - u^2/c^2)^{\frac{1}{2}} = 1/(1 - u'^2/c^2)^{\frac{1}{2}}.$$

These equations embody the idea of the Lorentz-Fitzgerald contraction, but with a marked difference to the original assumption. Originally, a rod was supposed to have an absolute maximum length when it is at rest in the ether, and this is its length for all observers whether they are at rest in the ether or not. If it is in motion through the ether in a direction parallel to its own length, its length shortens in the ratio $1 : (1 - u^2/c^2)^{\frac{1}{2}}$ where u is its velocity *through the ether*, and this length is again

is required in the reasoning. But in any case it would not be the observer B of the previous figure, but one nearer to O than B; i.e., B would be somewhat to the right of O′ in the figure.

independent of any observer's velocity. But Einstein's interpretation of the Lorentz equations assumes the length of a rod to be a relation between the rod and each particular observer; it varies from observer to observer. To an observer in relative rest with regard to it, the length attains its maximum value; and, theoretically, it may have any value between this and zero depending on the relative velocity of the observer to it, this result being quite independent of any velocity through the ether which may be attributed to it.

We have still to determine the relations between the y and z co-ordinates in the two frameworks. This is obtained very readily. Consider once more the emission of the flash of light from the common position of O and O′ at the common original instant, and the subsequent arrival of an element of the wave-front at a point P. Let d be the distance of P from its projection on the common axis as measured by the S group, and d' its distance as measured by the S′ group; and let x and x' be the respective abscissæ of the projection of P on this axis. Also let t and t' be the S and S′ times respectively of the arrival of the wave at P. Then by the second postulate

$$x^2 + d^2 = c^2t^2$$

and

$$x'^2 + d'^2 = c^2t'^2.$$

Now if the relations between x, x', t, and t' are assumed to be as before, it is easily proved that

$$x^2 - c^2t^2 = x'^2 - c^2t'^2.$$

Hence, to render the two equations just written consistent, we must assume also that $d = d'$. In other words, the length of any body measured transversally to the direction of relative motion is estimated to have the same value by the two groups.

Consequently our complete scheme of transformation is

$$\begin{aligned} x' &= \alpha(x - ut) \\ y' &= y;\ z' = z \\ t' &= \alpha(t - ux/c^2), \end{aligned}$$

or the equivalent scheme

$$\begin{aligned} x &= \alpha(x' - u't') \\ y &= y';\ z = z' \\ t &= \alpha(t' - u'x'/c^2), \end{aligned}$$

where $u' = -u$

and $\alpha = 1/(1 - u^2/c^2)^{\frac{1}{2}}.$

It will be obvious that just as length is a relation of a body

to an observer, so also will volume, superficial area and shape depend on the relative velocity of body to observer.

Suppose a body is moving with a uniform velocity u parallel to OX relative to the S frame of reference. If the dimensions of this body are to be measured by observers in this framework, they must contrive to obtain the co-ordinates of a sufficient number of points on its surface at the same instant, i.e., each point of the surface must have its co-ordinates measured at a definite value of the S time, say t units after the original instant. Let (x_1, y_1, z_1) and (x_2, y_2, z_2) be the values for two points P_1 and P_2 on the surface obtained in this way. To observers with respect to whom the body was fixed, these measurements would be a simpler operation, as they could naturally choose a framework in which each point would have invariable values for its co-ordinates. Let (x_1', y_1', z_1') and (x_2', y_2', z_2') be these fixed values for the points P_1 and P_2.

Then since

$$\begin{aligned} x_1' &= \alpha(x_1 - ut) \\ x_2' &= \alpha(x_2 - ut) \\ y_1' &= y_1; \quad z_1' = z_1 \\ y_2' &= y_2; \quad z_2' = z_2 \end{aligned}$$

it follows that

$$\begin{aligned} x_2' - x_1' &= \alpha(x_2 - x_1) \\ y_2' - y_1' &= y_2 - y_1 \\ z_2' - z_1' &= z_2 - z_1. \end{aligned}$$

Since α is greater than unity, $x_2' - x_1'$ is greater than $x_2 - x_1$, i.e., a linear dimension of a body as measured by observers to whom it is at rest contains a greater number of length units than the same dimension as measured by observers to whom it is in motion, provided the dimension is not directed at right angles to the relative motion of the body to the latter observers. It follows readily that the volume as measured by the observers to whom the body is at rest, is α times the volume as measured by the observers to whom it is in motion. This result is deduced by dividing the body into elementary parallelepipeda, with sides parallel and perpendicular to the direction of relative motion, and integrating the result for each element of volume. These considerations also involve a difference of shape as viewed by different observers. A body which is spherical to an observer to whom it is at rest would be an oblate spheroid to one to whom it is in motion, with its short axis parallel to the direction of motion. A body which is spherical to an observer to whom it is motion with a definite velocity would assume the form of a prolate spheroid to

observers to whom it is at rest, or travelling more slowly, and that of an oblate spheroid to observers to whom it is travelling more rapidly. It will be noted that the "Lorentz factor" a or $(1 - u^2/c^2)^{\frac{1}{2}}$, increases without limit as the value of u increases to c. Doubtless, so long as we are concerned with such relative speeds as occur in terrestrial events, or even in stellar movements, u is so small compared to c that a differs but little from unity. But conceivably the dimensions of a body might shrink without limit to those of a thin disc, provided its relative speed to the observer were sufficiently increased. It is also worthy of remark at this stage that if u were greater than c, a would become an imaginary quantity. Later it will appear that there are very weighty dynamical reasons for believing that relative speeds greater than that of light are physically impossible under the conditions postulated in Restricted Relativity.

These results concerning the relativity of shape and spatial dimensions are paralleled by a similar result concerning the relativity of intervals of time or "temporal dimension." Suppose an occurrence which lasts for some time takes place on a body which is in relative motion to a group of observers. Let us postulate another group of observers to whom the body is at rest. The first group employs a framework S, the second, S′. How will the first group measure the interval of time occupied by the occurrence? One of the group could do it provided light signals were despatched to him at the initial and final instants of the occurrence, and provided he made the usual allowances for the distances separating the body from himself at those instants. But this is tantamount to arranging two of the S group, each with a synchronised clock, so that one is adjacent to the body at the initial instant, and the other at the final instant. The difference of the two time-readings would give the measure of the interval which is valid for all the S group.

Idealising the body as a point for the purpose of mathematical expression, let it occupy a fixed position (x', y', z') in the S′ framework, and let an S′ observer also fixed there record the beginning and end of the occurrence as taking place at times t_1' and t_2' after the original instant.

The first of the two chosen observers of the S group will occupy a position (x_1, y_1, z_1) in his framework at the beginning of the occurrence, given by

$$x_1 = a(x' - u't_1')$$
$$y_1 = y' ; \ z_1 = z',$$

while the second of the two will be in a position (x_2, y_2, z_2) at the end of the occurrence, given by

$$x_2 = a(x' - u't_2')$$
$$y_2 = y' ; \; z_2 = z'.$$

But the important feature for our present purpose is not so much their positions as their time measurements of the initial and final instants. These are t_1 and t_2 respectively where

$$t_1 = a(t_1' - u'x'/c^2)$$
$$t_2 = a(t_2' - u'x'/c^2)$$

so that $$t_2 - t_1 = a(t_2' - t_1').$$

This result may be translated into words as follows :—

When an occurrence takes place in a locality, the measure of the time interval occupied by the occurrence, which is made by observers fixed at this locality, is the fraction $1/a$ or $(1 - u^2/c^2)^{\frac{1}{2}}$ of the measure of this interval, which is made by observers in relative motion to this locality with velocity u. Or we may put it in a more striking way. Let us take a pendulum as the material body of the previous reasoning, and a complete oscillation of it as the occurrence. Then it will be a quicker occurrence to an observer at rest beside it than to an observer moving past it, each one judging the time by clocks which were originally synchronised. Briefly, a clock in motion to an observer goes slower *for him* than a similar clock at rest to him.

These results concerning the relativity of our measures of spatial and temporal extension are obviously closely connected with the broader view which we have to adopt concerning simultaneity. Once more this is not an absolute property of events in themselves. Events may be simultaneous to one group of observers and not to another. This is the result which seems so paradoxical to the beginner, if he has not realised the nature of simultaneity in connection with events in different localities which do not come within the direct perception of one observer In illustrating this relativity of simultaneousness from the Lorentz equations as interpreted by Einstein, we can deduce a result more general than the previous two ; for the first of these deals with a purely spatial measurement at a given instant of time, while the second refers to a pure time measurement, the place of the measurement being fixed. Such operations are far greater abstractions than the ordinary man would be willing to admit ; all measurement

is much more a blend of space and time measurement than appears at the first glance, and the cogency of this view will be brought home to us as we proceed. Consider, therefore, the interval of time occupied by an occurrence which begins at one locality and ends at another, or, to put it in a manner suitable for mathematical analysis, the interval between an event (regarded as instantaneous) occurring at one point and an event occurring at another point.

The points P_1 and P_2 have co-ordinates (x_1', y_1', z_1') and (x_2', y_2', z_2') in the framework S′ in which they are fixed, so that these co-ordinates do not alter with lapse of time. One event occurs at P_1 at the instant t_1' units after the original instant, as measured by an observer at P_1 who is fixed in S′, the other occurs at the instant t_2', as measured by an observer fixed at P_2. (These events would be regarded as simultaneous by all observers in S′, if it transpired that $t_1' = t_2'$. That is the only criterion of simultaneity for this group of observers.)

How will these events be regarded by observers fixed in another framework S in relative uniform motion to S′? As before, any measurements of time made by observers in S will agree with measurements by two special S observers, of whom one is at P_1 at the moment the first event occurs there, and the second is at P_2 when the second event occurs there. The position of this first observer and his measure of the instant, in short, the S co-ordinates of the first event, are given by

$$\begin{aligned} x_1 &= a(x_1' - u't_1') \\ y_1 &= y_1' ;\ z_1 = z_1' \\ t_1 &= a(t_1' - u'x_1'/c^2). \end{aligned}$$

The S co-ordinates of the other event are given by

$$\begin{aligned} x_2 &= a(x_2' - u't_2') \\ y_2 &= y_2' ;\ z_2 = z_2' \\ t_2 &= a(t_2' - u'x_2'/c^2). \end{aligned}$$

Hence

$$\begin{aligned} t_2 - t_1 &= a(t_2' - t_1') + au(x_2' - x_1')/c^2 \\ &= a(t_2' - t_1' \pm uT/c^2), \end{aligned}$$

where T is the time required for light to travel a distance equal to the difference between x_1' and x_2'. (The double sign is necessary; we use the plus sign if $x_2' > x_1'$, and the minus sign if $x_2' < x_1'$.)*

* Remember that $u' = -u$, and u is essentially positive in the figures.

In general, therefore, if $t_1' = t_2'$, then $t_1 \neq t_2$, i.e., one event occurs at P_1 earlier or later than the other at P_2, *as viewed by* S *observers*.

It appears that if $x_2' > x_1'$, T is positive and $t_2 > t_1$. This means that if two events are simultaneous to observers who are fixed in the same framework as the places where the events happen, then for observers to whom the places are in uniform motion, the event happening at the place which is ahead of the other in its relative motion to these observers occurs later than the other.

Indeed, it is possible to deduce an apparently more paradoxical result. For it is quite possible that t_1' might be greater than t_2', and yet less than $t_2' + u\mathrm{T}/c^2$, and in such a case t_1 would be less than t_2 (adopting the upper sign), i.e., the order of the events in time would be opposite for the two groups of observers.

It should be noted that if $x_1 = x_2$, then $t_1' = t_2'$ implies $t_1 = t_2$. So if the relative motion of the one group to the other is perpendicular to the line joining the points, then events occurring at these points which are simultaneous to the one group are simultaneous to the other.

Before leaving this matter, we must once more take careful note that statements like these, which are such a blow to our preconceived notions of simultaneity, refer to events *happening at different places*. They do not refer to events happening at one place. In such a case we should equate x_1', y_1', z_1' to x_2', y_2', z_2' respectively, and so obtain the equation arrived at earlier:

$$t_2 - t_1 = a(t_2' - t_1').$$

This, of course, leads to a different measure of the interval between the events, as made by each group (a result dealt with above), but it does not disarrange their order, for

$$t_1 \lesseqgtr t_2$$

according as

$$t_1' \lesseqgtr t_2'.$$

We shall now deduce a result concerning angular measure, which will prove serviceable in the succeeding chapter.

Let the line P_1P_2 make an angle θ' with O′X′, then

$$\tan\theta' = (y_2' - y_1')/(x_2' - x_1').$$

(It is assumed, for convenience, that P_1 and P_2 are in the O′X′Y′ plane.)

P_1 and P_2 being fixed in the S′ framework, we know from our earlier results that their simultaneous positions (for S observers) in the S framework are connected with the S′ co-ordinates by the relations

$$(x_2' - x_1') = a(x_2 - x_1)$$
$$y_2' - y_1' = y_2 - y_1.$$

Hence, as the line P_1P_2 moves through the S′ framework, it remains constantly parallel to itself, making an angle θ with OX, given by

$$\tan \theta = (y_2 - y_1)/(x_2 - x_1) = a \tan \theta'.$$

It is readily seen from this result that the *acute* angle made by the moving line with the direction of its relative motion to the S observers is larger than the acute angle as measured by the S′ observers, to whom it is at rest. An exception, of course, arises when the angle is right, in which case $\theta' = \theta = \pi/2$.

In subsequent chapters we shall introduce a little simplification into the Lorentz equations of transformation by adopting as the unit of time, not the second, but the time in which light *in vacuo* travels over our unit of length. Thus if we employ the centimetre as the unit of length, the proposed unit of time would be $\frac{1}{3} \times 10^{-10}$ second approximately. This will involve writing u where we wrote u/c above; in fact, u will really stand for the ratio of the velocity of relative motion of S and S′ to the velocity of light; also, we shall write t where we previously wrote ct. By so doing we rid ourselves of the necessity of using a symbol, c, for the velocity of light. In these "relativity" units the equations of transformation become

$$\left.\begin{aligned} x' &= a(x - ut) \\ y' &= y\,;\ z' = z \\ t' &= a(t - ux) \end{aligned}\right\} \quad . \quad . \quad . \quad (5)$$

and

$$\left.\begin{aligned} x &= a(x' - u't') \\ y &= y'\,;\ z = z' \\ t &= a(t' - u'x') \end{aligned}\right\} \quad . \quad . \quad . \quad (6)$$

where $u = -u'$
and $a = 1/(1 - u^2)^{\frac{1}{2}}$.

The formal similarity of the x and t equations in each group is more marked than before.

It can easily be shown from (5) or (6) that

$$t^2 - x^2 - y^2 - z^2 = t'^2 - x'^2 - y'^2 - z'^2 \quad . \quad (7)$$

or, more generally, if two events are recorded in the frame S by the co-ordinates (x_1, y_1, z_1, t_1) and (x_2, y_2, z_2, t_2), and the same two events are recorded in the frame S′ by the co-ordinates (x_1', y_1', z_1', t_1') and (x_2', y_2', z_2', t_2'), then it can easily be proved that

$$(t_2 - t_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2$$
$$= (t_2' - t_1')^2 - (x_2' - x_1')^2 - (y_2' - y_1')^2 - (z_2' - z_1')^2 \quad (8)$$

Either member of (8) is a measure of the square of the "separation" between the events, and (8) is the mathematical expression of the fact that the separation is the same in all frames.

It will be observed that if we choose a frame S, such that $x_1 = x_2$, $y_1 = y_2$, $z_1 = z_2$, i.e., if the frame is such that the place of occurrence of the events is fixed relative to observers in this frame, then the separation is equal to the interval of time between the events as measured in this frame, the "proper" interval.

In the analysis of this chapter the axis of OX has a fictitious importance thrust upon it. This arises because of our choice of it as parallel to the relative direction of motion of the two frames. In a later chapter it will be shown that by abandoning this limitation a very elegant and symmetrical form can be given to the Lorentz transformation equations, still consistent, however, with the very important results (7) and (8).

CHAPTER II.

In the previous chapter we developed the Lorentz transformation equations on the Relativity hypothesis combined with the assumption of constant light velocity for all observers. In this chapter we shall apply these equations to certain kinematical problems, whose treatment on classical principles is familiar to all physicists.

If a body is moving with a uniform velocity relative to a frame S, it will also move with a uniform velocity relative to the frame S′, but the relation between the two velocities is by no means so simple as that obtained by the familiar method of compounding velocities.

Idealise the body as a particle, and suppose that it occupies the position (x, y, z) at the time t in the frame S and the position $(x + \delta x, y + \delta y, z + \delta z)$ at the time $t + \delta t$. Its velocity in the S frame has components v_x, v_y, v_z which are the limits of $\delta x/\delta t$, $\delta y/\delta t$, $\delta z/\delta t$.*

This same particle occupies a position (x', y', z') at the time t' in the frame S′ where

$$\begin{aligned} x' &= a(x - ut), \\ y' &= y\,;\ z' = z, \\ t' &= a(t - ux), \end{aligned}$$

and a position $(x' + \delta x', y' + \delta y', z' + \delta z')$ at the time $t' + \delta t'$, where

$$\begin{aligned} x' + \delta x' &= a\{(x + \delta x) - u(t + \delta t)\}, \\ y' + \delta y' &= y + \delta y\,;\ z' + \delta z' = z + \delta z, \\ t' + \delta t' &= a\{(t + \delta t) - u(x + \delta x)\}, \end{aligned}$$

so that

$$\left.\begin{aligned} \delta x' &= a(\delta x - u\delta t) \\ \delta y' &= \delta y\,;\ \delta z' = \delta z \\ \delta t' &= a(\delta t - u\delta x) \end{aligned}\right\} \quad . \quad . \quad . \quad (1)$$

By dividing the first three equations of (1) by the fourth we obtain

* It should be borne in mind that v is the ratio of the velocity of the particle to the velocity of light.

$$\left.\begin{aligned} v_x' &= (v_x - u)/(1 - uv_x) \\ v_y' &= v_y/a(1 - uv_x) \\ v_z' &= v_z/a(1 - uv_x) \end{aligned}\right\} \quad . \quad . \quad (2)$$

Similarly,

$$\left.\begin{aligned} v_x &= (v_x' - u')/(1 - u'v_x') \\ v_y &= v_y'/a(1 - u'v_x') \\ v_z &= v_z'/a(1 - u'v_x') \end{aligned}\right\} \quad . \quad . \quad (2A)$$

where v_x', v_y', v_z' are the three components of the velocity of the particle in the frame S′.

Later we shall see how these equations can be put in a form analogous to the equations by which we transform co-ordinates, but as they stand they are more suitable for developing a graphic method of compounding velocities in a manner consistent with the Relativity principle.

The familiar problem of relative motion is the following: There are three bodies K_1, K_2, K_3, such that the velocity of K_2 relative to K_1 (i.e., as measured from K_1) is known in direction and magnitude; also the velocity of K_3 relative to K_2 (as measured from K_2) is known. It is required to find the velocity of K_3 relative to K_1 (as measured from K_1).

The classical solution represents the velocity of K_2 relative to K_1 by a line AB, and the velocity of K_3 relative to K_2 by a line BC, and asserts that AC represents the resultant velocity or velocity of K_3 relative to K_1; and this would still be the correct solution even in Relativity kinematics were the two component velocities measured by the same observer or in the same frame. But in all cases of composition of velocities which are of any physical importance, this condition is not satisfied. To solve the problem above, we appeal to equations (2).

Suppose K_1 to be fixed in the frame S, and K_2 fixed in the frame S′, then if v_1 is the velocity of K_2 relative to K_1, and v_2 is the velocity of K_3 relative to K_2, $v_1 = u = -u'$, $v_2 = v'$, and the velocity of K_3 relative to K_1 is v.

We make use of (2A) and notice that

$$-u'v_x' = uv_x' = v_1 v_2 \cos\theta,$$

i.e., the so-called "geometric" product of the two velocities, or the product of one and the resolved part of the other along it. It will be convenient to denote it by the usual symbol of vector notation, viz.,

$$(\mathbf{v}_1 \,.\, \mathbf{v}_2).$$

Examination of the equations shows that the relative velocity of K_3 to K_1 is the ordinary vector sum or resultant of the three parts,

$$v_1/\{1 + (\mathbf{v}_1 . \mathbf{v}_2)\} \text{ parallel to the direction of } \mathbf{v}_1,$$

$$\frac{\text{the resolved part of } \mathbf{v}_2 \text{ parallel to } \mathbf{v}_1}{1 + (\mathbf{v}_1 . \mathbf{v}_2)},$$

and $$\frac{\text{the resolved part of } \mathbf{v}_2 \text{ perpendicular to } \mathbf{v}_1}{1 + (\mathbf{v}_1 . \mathbf{v}_2)} (1 - v_1^2)^{\frac{1}{2}}.$$

Hence it is easy to derive the construction given in Fig. 3. AB represents the velocity $\mathbf{v}_1$ and BC represents the velocity

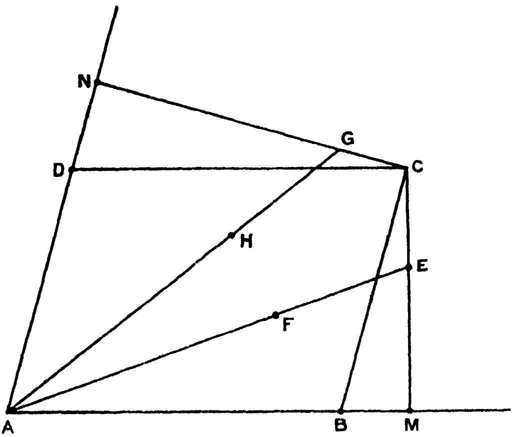

FIG. 3.—In the diagram AB represents a velocity equal to 0·8 the velocity of light; AD represents a velocity 0·6 that of light.

$$EM = (1 - (\cdot 8)^2)^{\frac{1}{2}} CM = \cdot 6\ CM$$
$$GN = (1 - (\cdot 6)^2)^{\frac{1}{2}} CN = \cdot 8\ CN$$
$$AF = AH = AE/1{\cdot}48 = AG/1{\cdot}48.$$

$\mathbf{v}_2$. From C drop a perpendicular CM on AB. Then the angle $CBM = \theta$.

Take a point E on CM, such that

$$EM = CM/a_1 = (1 - v_1^2)^{\frac{1}{2}}\ CM.$$

Join AE and take a point F on it such that

$$AF = AE/\{1 + (\mathbf{v}_1 . \mathbf{v}_2)\}.$$

Then AF represents the relative velocity of the body K_3 to K_1.

It is of vital importance for the appreciation of this result to keep in mind the fact that the two velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ are not measured by the one observer. It is forgetfulness of such important features which gives a bias to one's mind, accustomed to the older ideas, to regard a result such as this as a paradox. Incidentally it will be noticed that the angle θ is measured in the frame S′ relative to which K_2 is at rest.

It will be a serviceable exercise if we interchange the velocities, and suppose that K_2 moves relative to K_1 with a velocity $\mathbf{v}_2$ and K_3 relative to K_2 with a velocity $\mathbf{v}_1$.

The velocity of K_3 relative to K_1 is not the same as before. The construction would require us to take the other two sides of the parallelogram ABCD, drop a perpendicular CN on BD, and take a point G on it such that

$$\mathrm{CG} = \mathrm{CN}/\alpha_2 = (1 - v_2^2)^{\frac{1}{2}}\,\mathrm{CN}.$$

Choosing H on AG, so that

$$\mathrm{AH} = \mathrm{AG}/\{1 + (\mathbf{v}_1 \cdot \mathbf{v}_2)\},$$

it appears that AH represents the required relative velocity.

This latter construction can be viewed in another way, in connection with the first arrangement of the velocities, where K_2 was moving with velocity $\mathbf{v}_1$ relative to K_1, and K_3 with velocity $\mathbf{v}_2$ relative to K_2. Reversing this we should say that K_2 was moving relative to K_3 with a velocity equal and opposite to $\mathbf{v}_2$, and K_1 relative to K_2 with a velocity equal and opposite to $\mathbf{v}_1$. On carrying out the construction, starting from the point C, it is easy to see that the velocity of K_1 relative to K_3 is represented by a line equal and parallel to HA (in the sense of the letters as written).

This is another of those results which appear paradoxical if one forgets the fact that the relative velocity of K_3 to K_1 is measured in a frame in which K_1 is fixed, while the relative velocity of K_1 to K_3 is measured in a frame in which K_3 is fixed. It appears that the observers do not "get the same line for their relative motion" (as it was once put to the author in a crude fashion by a critic). More precisely, angles between direction of relative motion and other definite lines are not measured alike by two groups of observers. But the result which was demonstrated at the end of the previous chapter has prepared us to accept such a conclusion as no paradox, but quite in accordance with Relativity ideas.

There is, of course, one thing which is measured alike, and

that is the *magnitude* of the relative velocities of K_3 to K_1 and of K_1 to K_3. A contrary result could hardly be anticipated, in view of the fact of which we have made constant use in the treatment of the S and S′ frames, that u and u' have the same numerical value. In the present instance the result is easily proved; for

$$\begin{aligned} AE^2 &= (v_1 + v_2 \cos \theta)^2 + (1 - v_1^2)v_2^2 \sin^2\theta \\ &= v_1^2 + v_2^2 + 2v_1v_2 \cos \theta - (v_1v_2 \sin \theta)^2, \end{aligned}$$

and AG^2 can be shown to be equal to the same function of v_1 and v_2. Hence, since AE is equal to AG, so also is AF equal to AH.

In short, although the order of addition of two velocities makes no difference to the magnitude of the resultant, it does make a difference to its direction. But, to reiterate the important fact, it is implied that different observers are involved in the measurement of the component velocities. If one observer wanted to add the two velocities of two bodies both measured in his own frame, he would do so in the familiar way by the parallelogram method, but as far as the author is aware, such a result has no physical significance whatever.

In the figure the velocities are pictured as making an acute angle one with the other. Were the angle obtuse, $(\mathbf{v}_1 . \mathbf{v}_2)$ $(= v_1v_2 \cos \theta)$ would be negative, $1 + (\mathbf{v}_1 . \mathbf{v}_2)$ would be less than unity, and AF would be longer than AE. If the angle were right E and F would coincide.

An interesting case arises when $\mathbf{v}_1$ and $\mathbf{v}_2$ have the same direction, so that $(\mathbf{v}_1 . \mathbf{v}_2) = v_1v_2$. The magnitude of the resultant velocity is then

$$(v_1 + v_2)/(1 + v_1v_2) \quad . \quad . \quad . \quad (2B)$$

As a consequence the resultant has a magnitude less than the numerical sum of the components. It is interesting in this connection to recall the fact that the introduction of velocities greater than that of light would introduce imaginary factors into the equations of transformation. A glance at the formula just written will show that the composition of two velocities less than that of light will produce a resultant with a magnitude less than the speed of light, for if v_1 and v_2 are each less than unity, so also is $(v_1 + v_2)/(1 + v_1v_2)$ less than unity. If v_1 is equal to unity, the resultant is unity no matter what value v_2 has, or the addition of the velocity of light to *any* velocity, equal to, greater than or less than that of light produces a resultant equal to that of light. This is true also if the velocities

are inclined at a finite angle to each other. Reference to Fig. 3 will show that if $v_1 = 1$, then E coincides with M, and F with B. If anyone were bent on mere mystification, he could make a great feature of the result that with both velocities greater than that of light the resultant would be less than that of light; for, to be sure, if v_1 and v_2 be individually greater than unity, $(v_1 + v_2)/(1 + v_1 v_2)$ is less than unity, and decreases to zero as a limit as v_1 and v_2 increase without limit. But this is mere symbol juggling, because in assuming velocities greater than that of light we have outstepped our conditions of transformation, which accepted light signals as the suitable means of communication between the two frames for the purpose of establishing relations between physical measures. As already stated, we shall become acquainted later with dynamical reasons for concluding that velocities greater than that of light cannot occur *under the conditions postulated in restricted Relativity.*

We can now return to the equations (2), for the purpose of expressing them in a rather more convenient form. It will be advisable at this stage to introduce once more the concept of the "proper time" of the particle, the term introduced by Minkowski, who was apparently the first to realise its importance. By squaring equations (1), we can easily show that

$$\delta t'^2 - \delta x'^2 - \delta y'^2 - \delta z'^2 = \delta t^2 - \delta x^2 - \delta y^2 - \delta z^2 \qquad (3)$$

and hence that

$$\delta t'(1 - v'^2)^{\frac{1}{2}} = \delta t(1 - v^2)^{\frac{1}{2}}.$$

Supposing, therefore, in the frame S, we integrate the expression $(1 - v^2)^{\frac{1}{2}}dt$ along the path of a particle between two defined positions, the result will be the same as a similar integration will yield when carried out in any other frame moving with a uniform velocity with respect to S. Such a quantity which retains the same value in all the frames we refer to as "invariant." The particular invariant in question is called an interval of "proper time" for the particle. Its successive elements might be arrived at thus. During any element of its path, the particle will be momentarily at rest in some one of the set of frames S, S′, etc. Since in that frame the v is zero at the moment, the actual element of time measured in that frame will be an element of the proper time of the particle; for the factor $(1 - v^2)^{\frac{1}{2}}$ is unity if $v = 0$. Hence if we denote the proper time of a particle from any arbitrary position on

its path as τ, then

$$\delta\tau = \delta t(1 - v^2)^{\frac{1}{2}} = \delta t'(1 - v'^2)^{\frac{1}{2}} = \text{etc.}$$

It will be convenient to introduce the symbol β to represent the quantity $1/(1 - v^2)^{\frac{1}{2}}$, which is, as it were, the Lorentz factor for the S frame and the (momentary) " rest " frame of the particle ; so that

$$\delta t/\beta = \delta\tau$$

is an elementary magnitude which is invariant for all the frames.* It will be seen that the proper time of the particle between two positions is also the separation between the events which are its passage through the first and its passage through the second position.

Dividing each side of equations (1) by $\delta\tau$ and going to the limit, we have

$$\begin{aligned} dx'/d\tau &= \alpha(dx/d\tau - u\,dt/d\tau) \\ dy'/d\tau &= dy/d\tau\ ;\ dz'/d\tau = dz/d\tau \\ dt'/d\tau &= \alpha(dt/d\tau - u\,dx/d\tau). \end{aligned}$$

These equations imply that x, y, z, t and x', y', z', t' are conceived to be functions of the invariant proper time τ measured from a definite point of the path of the particle, and we see that the four differential coefficients of the four co-ordinates of position and time with respect to the proper time are transformed from frame to frame just as the co-ordinates themselves are. It will be convenient to introduce the fluxion notation of dots to indicate differential coefficients with respect to the *proper time*. After a little practice, it becomes quite easy to guard against confusing the dot with the operator d/dt (or, as it really should be written, $\partial/\partial t$, since in this connection t is no longer an independent variable, but just on the same footing as x, y, z, viz., a variable depending on τ). So we re-write the equations just developed thus :

$$\left.\begin{aligned} \dot{x}' &= \alpha(\dot{x} - u\dot{t}) \\ \dot{y}' &= \dot{y}\ ;\ \dot{z}' = \dot{z} \\ \dot{t}' &= \alpha(\dot{t} - u\dot{x}) \end{aligned}\right\} \quad . \quad . \quad . \quad (4)$$

The quantities $\dot{x}$, $\dot{y}$, $\dot{z}$ have the dimension of velocity, but they are not the velocities of the particle in any definite frame.

* No confusion should arise between α and β. u and α are used in connection with the relative velocities of two frames while v and β are used in connection with the velocity of a body relative to a frame.

They are a sort of "blended" velocity obtained by using the elementary displacement of the particle in one frame with the corresponding element of time as measured in the (momentary) "rest" frame of the particle.

It is very easy, however, to express these dotted co-ordinates in terms of the velocity in a given frame; for

$$\delta x/\delta\tau = \beta\delta x/\delta t$$

and so

$$\begin{aligned} \dot{x} &= \beta v_x, \\ \dot{y} &= \beta v_y, \\ \dot{z} &= \beta v_z, \end{aligned}$$

and, of course,

$$\dot{t} = \beta.$$

Hence we can write the transformation equations for velocities in the form

$$\left.\begin{aligned} \beta' v_x' &= \alpha(\beta v_x - u\beta) \\ \beta' v_y' &= \beta v_y\,;\ \ \beta' v_z' = \beta v_z \\ \beta' &= \alpha(\beta - u\beta v_x) \end{aligned}\right\} \quad . \quad . \quad (4\text{A})$$

showing that

$$\beta v_x,\ \beta v_y,\ \beta v_z,\ \beta$$

or

$$v_x/(1 - v^2)^{\frac{1}{2}},\ v_y/(1 - v^2)^{\frac{1}{2}},\ v_z/(1 - v^2)^{\frac{1}{2}},\ 1/(1 - v^2)^{\frac{1}{2}}$$

transform like the co-ordinates, or are "cogredient" with the co-ordinates.

When we come to the discussion of dynamical principles on the Relativity hypothesis, we shall have to deal with the accelerations of bodies in a frame, and shall require the transformation equations for the accelerations of a given body in different frames. The explicit form of these equations is somewhat cumbersome, but they are implicitly contained in equations which are directly derived from (4) by a further differentiation with respect to the proper time. For just as differentiation of the four space-time co-ordinates of a particle with respect to the proper time gives a tetrad of quantities cogredient with the co-ordinates, i.e., subject to the Lorentz transformation, so a differentiation of this tetrad with respect to the proper time will give a further tetrad of cogredient quantities. Indeed, it is this property which makes proper time such a very serviceable concept, viz., the derivation by differentiation with respect to it of successive tetrads of quantities cogredient with the co-ordinates, and this property clearly depends on the invariance of the element of proper time, no matter the frame in which one elects to measure it.

Differentiation of (4) yields

$$\left.\begin{aligned} \dot{x}' &= \alpha(\dot{x} - u\dot{t}) \\ \dot{y}' &= \dot{y};\ \dot{z} = \dot{z} \\ \dot{t}' &= \alpha(\dot{t} - u\dot{x}) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (5)$$

which, as stated, contains implicitly the relations between accelerations in each frame of the same particle; of course, $\ddot{x}$, $\ddot{y}$, $\ddot{z}$, or $d^2x/d\tau^2$, etc., are not acceleration components in any frame, any more than $\dot{x}$, $\dot{y}$, $\dot{z}$ are velocity components in a frame. But it is easy to express them in terms of the true accelerations in a frame. To begin with

$$\begin{aligned} \ddot{t} = d\beta/d\tau = \beta d\beta/dt &= \beta d(1 - v^2)^{\frac{1}{2}}/dt \\ &= \tfrac{1}{2}\beta dv^2/dt \div (1 - v^2)^{\frac{3}{2}} \\ &= \beta^4(v_x a_x + v_y a_y + v_z a_z) \quad . \quad . \quad . \quad (6) \end{aligned}$$

where a_x, a_y, a_z are the components of the acceleration, for

$$dv^2/dt = d(\Sigma v_x^2)/dt = \Sigma(v_x dv_x/dt)$$

The bracketed expression in (6) is the geometric product of velocity and acceleration, and may for convenience be denoted in future by the usual vectorial symbol $(\mathbf{v}\,.\,\mathbf{a})$, so that

$$\ddot{t} = \beta^4(\mathbf{v}\,.\,\mathbf{a}) \quad . \quad . \quad . \quad (6\text{A})$$

Turning now to x, y, z, we have

$$\begin{aligned} \ddot{x} &= \beta d(\beta v_x)/dt \\ &= \beta^2 dv_x/dt + \beta v_x d\beta/dt \\ &= \beta^2\{a_x + \beta^2 v_x(\mathbf{v}\,.\,\mathbf{a})\} \end{aligned}$$

similarly,

$$\begin{aligned} \ddot{y} &= \beta^2\{a_y + \beta^2 v_y(\mathbf{v}\,.\,\mathbf{a})\} \\ \ddot{z} &= \beta^2\{a_z + \beta^2 v_z(\mathbf{v}\,.\,\mathbf{a})\} \quad . \quad . \quad . \quad (7) \end{aligned}$$

If we write b to denote the quantity $\beta^2(\mathbf{v}\,.\,\mathbf{a})$, we see that the four functions of velocity and acceleration,

$$\beta^2(a_x + bv_x),\ \beta^2(a_y + bv_y),\ \beta^2(a_z + bv_z),\ \beta^2 b$$

are cogredient with the co-ordinates. Hence for the purpose of transforming accelerations from the frame S to the frame S′, we have as a convenient form of (5) (or rather (5) with the systems interchanged, i.e., the equations reciprocal to (5)) :

$$\left.\begin{aligned} \beta^2(a_x + bv_x) &= \alpha\beta'^2\{a_x' + b'(v_x' - u')\} \\ \beta^2(a_y + bv_y) &= \beta'^2(a_y' + b'v_y') \\ \beta^2(a_z + bv_z) &= \beta'^2(a_z' + b'v_z') \\ \beta^2 b &= \alpha\beta'^2\{b'(1 - u'v_x') - u'a_x'\} \end{aligned}\right\} \quad . \quad (8)$$

Equations (8) can be applied to a special case of some importance. Suppose the frame S′ to be the momentary rest frame of the particle, so that $v' = 0$, then the acceleration as measured in that frame is called the "rest-acceleration." (The concept has been used in pre-Relativity days in connection with what has been called the "rest-mass" of an electron, i.e., the quotient of the force exerted on it by its acceleration when its velocity is vanishingly small.) Since S′ is the rest-frame of the particle at the moment, it follows that

$$v' = 0,\ \beta' = 1,\ \text{and } b' = 0.$$

Also,

$$\begin{aligned} v &= v_x,\ \text{since } v_y = \beta' v_y'/\beta = 0 = v_z \\ u' &= -u = -v \\ a &= \beta \end{aligned}$$

and

$$b = \beta^2 v a_x.$$

Hence by (8)

$$\begin{aligned} \beta^2 a_x + \beta^4 v^2 a_x &= \beta a_x' \\ \beta^2 a_y &= a_y' \\ \beta^2 a_z &= a_z' \\ \beta^4 v a_x &= \beta v a_x' \end{aligned}$$

which reduce to

$$\begin{aligned} \beta^3 a_x &= a_x' \\ \beta^2 a_y &= a_y' \\ \beta^2 a_z &= a_z' \end{aligned}$$

(the first and fourth of the above being identical), or

$$\left.\begin{aligned} a_x &= a_x'(1 - v^2)^{\frac{3}{2}} \\ a_y &= a_y'(1 - v^2) \\ a_z &= a_z'(1 - v^2) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (9)$$

It should be noted that our choice of axes makes OX parallel to the direction of relative motion of S and S′. Hence that axis must also be parallel to the direction of motion of the particle at the moment in S. So we can express the result (9) independently of any axes by stating that

the tangential acceleration in S $= (1 - v^2) \times$ tangential rest-acceleration,

and the normal acceleration in S $= (1 - v^2)^{\frac{3}{2}} \times$ normal rest-acceleration.

It is one of the first exercises presented to a beginner in kinematics to study the rectilinear motion of a body with uniform acceleration. It is not without interest to work out

corresponding results for the corresponding problem in Relativity kinematics, viz., the rectilinear motion of a body which has a uniform *rest-acceleration* in a frame S. Write g for this constant value, and assume that the motion is parallel to the axis OX. Then in (9) we put a_y' and a_z' equal to zero, and $a_x' = g$.

Hence

$$a_y = a_z = 0$$

and

$$a_x = g(1 - v^2)^{3/2},$$

i.e.,

$$dv/dt = g(1 - v^2)^{3/2} \qquad . \quad . \quad . \quad (10)$$

Integrating, we obtain

$$v/(1 - v^2)^{1/2} = gt,$$

choosing the origin of time to be the instant at which the body began to move.

Hence

$$dx/dt = v = gt/(1 + g^2t^2)^{1/2} \qquad . \quad . \quad (11)$$

Another integration yields

$$gx = (1 + g^2t^2)^{1/2}$$

if we choose the origin O to be such that the body began to move from the point $x = 1/g$.

Since

$$x^2 - t^2 = (1/g)^2 \qquad . \quad . \quad . \quad (12)$$

it is clear that if one constructs a distance-time graph of the motion in the usual way, the graph is not the parabola of the uniformly accelerated motion of classical kinematics, but an equilateral hyperbola, whose semi-axis is $1/g$, and therefore varies inversely as the value of the rest-acceleration.

It is of interest to note that by (10) the value of the actual acceleration in the frame S decreases to zero as v approaches unity (i.e., the velocity of light). Equation (11) also shows that unity is the limit of v as t approaches infinity; and (12) shows that as x and t approach infinity, their difference approaches zero, a fact which is also embodied in the graph, for at great distances along either branch of a hyperbola there is approximate coincidence of the curve and its asymptotes.

The following three results are easily verified:

$$\left.\begin{aligned} t &= (g^2x^2 - 1)^{1/2}/g \\ v &= (g^2x^2 - 1)^{1/2}/gx = t/x \\ a &= 1/g^2x^3 \end{aligned}\right\} \qquad . \quad . \quad . \quad (13)$$

It is not unnatural for anyone glancing at these equations

for the first time to be a little puzzled by their apparent lack of consistency with the well-known rules concerning dimensions of physical quantities. (The second of (13), e.g., looks " upside-down.") This arises, of course, from our special choice of units. It is clear that since time is expressed in units which are $1/c$ of a second, accelerations will be expressed in units which are c^2 times larger than the usual C.G.S unit of acceleration. The simple " dodge " for converting all our kinematic formulæ in relativity units into formulæ in ordinary (C.G.S.) units is to replace

t in the relativity formula by ct,
u and v ,, ,, ,, ,, u/c, v/c,
a and g ,, ,, ,, ,, a/c^2, g/c^2.

Hence in *usual* units the results (10), (11), (12), and (13) above would become

$$dv/dt = g(1 - v^2/c^2)^{\frac{3}{2}} \quad . \quad . \quad . \quad (10\text{A})$$

$$v = cgt/(g^2t^2 + c^2)^{\frac{1}{2}} \quad . \quad . \quad . \quad (11\text{A})$$

$$x^2 - c^2t^2 = (c^2/g)^2 \quad . \quad . \quad . \quad . \quad (12\text{A})$$

$$\left.\begin{aligned} ct &= (g^2x^2 - c^4)^{\frac{1}{2}}/g \\ v &= c^2t/x \\ a &= c^6/g^2x^3 \end{aligned}\right\} \quad . \quad . \quad . \quad (13\text{A})$$

The constantly recurring feature of Relativity that the velocity of light is a limit for all physically possible velocities leads us to make note of a fact which, in a generalised form, plays a very important part in gravitational theory It is clear that for a given differential increment of the time in a frame S, the corresponding increment of proper time of a body decreases steadily to zero as the speed of the body approaches that of light, for

$$\delta\tau = \delta t(1 - v^2)^{\frac{1}{2}}.$$

This is simply pushing to its logical conclusion the result obtained in the previous chapter, that events taking place in a frame S′ appear to " go slow " to observers in S, and to " slow up " more and more as the relative speed of S′ to S increases ; so much so that if the limiting velocity of light were attained, there would be no progression of events in S′ at all as far as observers in S could see, assuming that they could see at all in such wildly impracticable circumstances. Professor Eddington puts this in a picturesque way when he says : " If man wishes to achieve immortality and eternal youth, all he has to do is to cruise about space with the velocity of light.

He will return to earth after what seems to him an instant to find many centuries passed away."

This can also be seen by reference to the result proved earlier in the chapter, that addition of any velocity to the velocity of light in a given direction alters neither the magnitude nor direction of this latter velocity; which means that if a body had the speed of light relative to an observer in a given direction, all bodies moving about over this body would have the same velocity to this observer, i.e., would appear to be at rest on it to this observer. So " nothing would happen " on the body *for this observer*.

APPENDIX TO CHAPTER II.

As pointed out in the first chapter, although optical effects depending on the motion of an observer through the ether have not been rendered evident in physical experiments, there are a number of such effects which depend on the relative motion of the observer and the source of light. Three of these can be very readily treated by means of the formulæ and results developed in the last two chapters.

Consider, for example, an observer situated near the origin of the frame S, viewing a distant source of light situated at a point P, such that the direction cosines of OP are l, m, n. The part of the wave train which reaches this observer is practically a plane wave with direction of propagation, $-l$, $-m$, $-n$, and the light vector is a harmonically varying magnitude which varies as the expression

$$\sin p(t + lx + my + nz)$$

where p is the pulsance ($2\pi \times$ frequency) of the light, assumed to be monochromatic; the frequency is, of course, expressed in vibrations per $1/c$ second. Now consider the frame S′ and an observer fixed in it, also near its origin, and let us elicit how he would measure the same element of the wave train.* We can do so by substituting for x, y, z, t their values in terms of x', y', z', t'. This shows us that the S′ observer finds the light vector varying as the harmonic function

$$\sin p[a(t' - u'x') + la(x' - u't') + my' + nz'],$$

i.e.,

$$\sin p'(t' + l'x' + m'y' + n'z')$$

where

$$\left.\begin{aligned} p' &= a(1 + lu)p \\ l' &= (l + u)/(1 + lu) \\ m' &= m/a(1 + lu) \\ n' &= n/a(1 + lu) \end{aligned}\right\} \quad . \quad . \quad . \quad (1)$$

The last three of these equations embody the well-known

* We are assuming the observations to be made when the two origins are near one another.

aberration of light. The frame S is supposed to be fixed with respect to the sun and the frame S′ to the earth. For a short interval of time such frames could be regarded as in uniform relative motion, u being the speed of the earth in its orbit. The angular co-ordinates of a distant star, as measured by an observer on the earth and an imaginary observer fixed near it on the earth's orbit, would not agree; their cosines would be related as indicated in equations (1). In the particular case when the star is situated, say, on the axis OZ, so that in the frame S the light rays are perpendicular to the direction of motion of S′ relative to S, the result becomes very simple. Since $l = m = 0$ and $n = 1$, we have

$$l' = u, \; m' = 0, \; n' = 1/a.$$

So the light makes an angle θ' with Z′O′ in the frame S′, such that

$$\tan \theta' = au.$$

Classical theory yields in this case the result that the angle of aberration is arc tan u. As u is of the order 1/10,000, and therefore a about $1 - 10^{-8}$, astronomical observations are incapable of deciding between the two results.

But there is one difficulty in aberration theory which Relativity surmounts readily—very naturally so, since its object was to give a comprehensive explanation of this and similar difficulties. As long ago as the middle of the eighteenth century Boscovich had suggested the use of a telescope filled with water in aberration experiments. His suggestion was never carried out until Airy in 1871 made some observations on the star γ Draconis with such a telescope, at the Greenwich observatory. But in the interval of a century there had been a very searching discussion of the possible results of such an experiment. Some physicists believed that owing to the slower speed of the light through the water, the angle of aberration as measured by such a telescope would be larger than that measured by an ordinary telescope whose tube contained air, the relation between the two measures depending on the index of refraction for air and water. Fresnel, however, in endeavouring to discover an explanation for an experiment performed by Arago, was led to propound a hypothesis concerning what is called in theories of the ether the "dragging coefficient of matter on ether," and on applying this hypothesis to Boscovich's hypothetical experiment, predicted that the material filling the telescope would not affect the angle of aberration. This was

in 1818. But although in the meanwhile Fizeau had (in 1851) tested, with quite satisfactory results, Fresnel's hypothesis by another experiment, to which we shall refer presently, it was not until 1871 that his prediction concerning the aberration as measured by a water-filled telescope was verified by the work of Sir G. Airy. Of course, this independence of the angle of aberration and material filling the telescope tube is just what would be expected on the relativity view. The direction cosines of the light rays in the S′ frame are $-l'$, $-m'$, $-n'$, and are determined entirely by the cosines in the S frame and the relative velocity of S′ and S. If an observer wishes to focus an image of the star on the cross-wires of any symmetrical optical instrument, he naturally will lay the axis of the instrument along l', m', n', no matter what transparent media fill the space between the lenses or mirrors of the instrument.

Returning to equations (1), we deduce that the frequency of the light in the S′ frame is not the same as that in the S. This is the familiar Döppler principle, but with some variations from the customary result. Let us take two special cases so as to elucidate the difference.

If the light is being propagated opposite to the relative motion of S′ to S, then $l = 1$ and $m = n = 0$.

Hence

$$p' = \alpha(1 + u)p = p(1 + u)^{\frac{1}{2}}/(1 - u)^{\frac{1}{2}}.$$

If the propagation is in the same direction as the relative motion of S′ to S, $l = -1$, $m = n = 0$, and

$$p' = \alpha(1 - u)p = p(1 - u)^{\frac{1}{2}}/(1 + u)^{\frac{1}{2}}.$$

Now classical ether theory yields somewhat different results. Thus it would discriminate between the case where the source is stationary in the ether and the observer moving towards it with a velocity u, and the case where the observer is at rest and the source approaching him, in the first case giving $p(1 + u)$ as the altered frequency, and $p/(1 - u)$ in the second. Relativity theory makes no such discrimination, and gives a value $p(1 + u)^{\frac{1}{2}}/(1 - u)^{\frac{1}{2}}$ intermediate to the previous two. As the three results only differ in squares and higher powers of u, observation is not sufficiently refined to give preference to one more than another. When the observer and source are receding from one another, the corresponding values are

$$p(1 - u),\ p/(1 + u),\ p(1 - u)^{\frac{1}{2}}/(1 + u)^{\frac{1}{2}}.$$

If we also consider a beam of light propagated at right

angles to the direction of relative motion, so that $l = o$, then

$$p' = ap,$$

or there would be altered frequency, the difference being of the second order in the ratio of the velocities of relative motion and light. As far as the author is aware, no verification of this result has been attempted, although it has been suggested that with great care a definite result might be obtained by analysis of the light emitted by the positive rays in directions perpendicular to the cathode channel.

The third effect coming under review is the so-called "dragging effect" of moving transparent media on light, referred to above.

The propagation of light in a material medium, which is itself in motion relative to the source and the observer's instruments, is a problem which has evoked a great deal of discussion and controversy even in the days of the elastic-solid theory of the ether. When optical theory became merged in general electromagnetic theory, the matter became still more involved; and, indeed, it was the labours of Lorentz and Larmor to evolve order out of this mass of uncorrelated material that led directly to the position in which the formulation of an exact relativity principle became possible. When the concept of the ether as a medium of propagation was ousting the corpuscular theory from the position of privilege which it had occupied in works on natural philosophy, there were two extreme hypotheses regarding propagation in moving matter which were equally plausible. One was that the *motion* of the matter produced no effect on the propagation of the light through the ether, i.e., to choose a definite case, if the velocity of light in the ether is c cms. per sec. *in vacuo*, and therefore c/μ cms. per sec. in a transparent body at rest in the ether (μ being the refractive index), then the velocity *relative to the ether* is still c/μ cms. per sec., even if the body is moving through the ether, and so is $c/\mu - v$, or $c/\mu + v$, cms. per sec. *relative to the body*, according as the motion of the body is in the same or in the opposite direction to that of the light. The other extreme view regarded the light as travelling with the same velocity with regard to the body, c/μ, as when at rest; but with a velocity $c/\mu + v$, or $c/\mu - v$, with regard to the ether. On the former view the body was considered to have no "dragging effect" on the ether, while on the latter it was conceived to have a "coefficient of convection" equal to unity. Owing to the special views which Fresnel entertained concerning

the elasticity and density of the ether *in vacuo* and in matter, he was led to propound a view intermediate to these extremes. He proposed a "convection-coefficient" equal to $1 - 1/\mu^2$, i.e., a velocity relative to ether of $c/\mu + v(1 - 1/\mu^2)$, and to the body of $c/\mu - v/\mu^2$, where the body and light move in the same direction; by means of this formula he succeeded in accounting for an experimental result which had just been communicated to him by Arago, and at the same time predicted the independence of aberration to the material filling the telescope, a prophecy verified fifty years later by Airy, as recounted. In 1851 Fizeau made what seems, at the first glance, to be a direct verification of Fresnel's hypothesis, but is really an illustration of a purely relative phenomenon with no *direct* bearing on the ether question at all. What Fizeau recorded (and his work was repeated with further refinements in 1886 by Michelson and Morley, with quantitative results much more precise) was that light travelling in flowing water had, relative to the pipe in which the water flowed, and to the observer's apparatus stationed at one end of the pipe, a velocity neither c/μ nor $c/\mu \pm v$, but a velocity $c/\mu \pm \kappa v$, where κ is a fraction less than unity. The comparison of κ with $1 - 1/\mu^2$, effected by the work of the two American physicists, was very satisfactory. On account of the agreement of Fresnel's convection-coefficient with experiment, it became in later years a desideratum for any form of electromagnetic theory to deduce this coefficient, at all events, to an accuracy of the order of v/c, if it expected to command attention. Indeed, so long as effects of the first order only were in question, this simple result of Fresnel's was quite sufficient to account for the failure of velocities relative to the ether to condescend so far as to allow themselves to be measured. It was only when Michelson and Morley's "ether-drift" experiment showed that this lack of condescension extended even to second-order effects (depending on v^2/c^2), that Fresnel's coefficient became inadequate as an excuse for the ether's bashfulness. To those interested in the historical development of pre-relativity views, accounts are obtainable in standard English text-books on Optics, or in Whittaker's "History of Theories of the Aether." It is interesting to note that the theory of a *mobile* ether, which played an important part at one time and was developed by Stokes, but was abandoned by later physicists, has been recently reintroduced into discussion (with no great conviction on the author's part apparently) by Dr. Silberstein. In a paper to the February "Phil. Mag." of 1920, he points out that recent results on the

velocity of light near gravitating matter gives some plausibility to an hypothesis concerning condensation of the ether near celestial bodies, which Planck suggested some years ago as a necessary addendum to Stokes' theory. To those who wish to consider " a last glimpse of hope for the banished medium," a perusal of this paper will prove interesting and helpful.

To return, however, to Fizeau's experiment, the reader will see that in it (as in all experimental work on this vexed question yielding so-called positive results) there is no direct question of ethereal velocity involved at all. Here, as always, measured velocities are velocities relative to matter—in this case, the water and the pipe. The treatment of this problem on the relativity basis is an excellent illustration of Einstein kinematics. The water is at rest in a frame S′, and the velocity of light through it is $1/\mu$ (the velocity *in vacuo* being unity); the frame S′ is in motion with the velocity u along OX, relative to S. The light being propagated in S′ along O′X′, its velocity relative to S, which is supposed to hold the pipe and observer's apparatus, is obtained from equation (2B) of this chapter by putting v_1 equal to u, and v_2 equal to $1/\mu$. The result is

$$(u + 1/\mu)/(1 + u/\mu).$$

If we neglect squares and higher powers of u, this is approximately equal to

$$\begin{aligned} &(u + 1/\mu)(1 - u/\mu) \\ &= 1/\mu + u(1 - 1/\mu^2), \end{aligned}$$

which is Fresnel's result.

This note would be incomplete, however, if it failed to emphasise once more the totally different standpoints of Relativity and earlier theory. On the basis of a fixed ether, physicists deduced that certain optical effects depending on relative motion of matter and ether should manifest themselves. Failure on the part of these to exhibit themselves led to the introduction of additional hypotheses such as that of a convection-coefficient for first-order phenomena, and a contraction-coefficient for second-order, which produced agreement between observation and theory, so long as observation could not be pushed beyond a certain degree of precision—a not altogether satisfactory state of affairs. Relativity accepts the absence of these expected phenomena as an exact and fundamental natural law, and not as the result of a peculiar conspiracy on the part of matter to hide the blushing ether from our view. It treats the matter in its own particular and mathematically

simple way, and finds formulæ which naturally imply these negative results, and which the thorough-going relativist accepts as exact and verifiable by any experiment conceivable, however precise.

There is a celebrated experiment carried out by Sagnac which should perhaps be mentioned here, although it has no real bearing on the restricted Relativity which we are now discussing.* It used, however, to be referred to as in contradiction to the hypothesis of Relativity. A pencil of light issuing from a source is divided in two by the usual devices and one-half is made to travel round the perimeter of a circular disc in one sense by means of mirrors suitably placed and tangential to the circumference, the other half being sent the other way. The two partial beams, after their respective journeys, meet once more and interfere to produce a set of fringes. The disc is now rotated, and there is a displacement of the fringes.

Now as the disc had a radius of 25 cms., and was turned only at the rate of two revolutions per second or thereabouts, it should be obvious even to a beginner that the *second-order* effects, with which Relativity is primarily concerned in such experiments, have no place here at all. The result is purely a *first-order* effect, and the reader will find an elegant treatment of it as such in the "Comptes Rendus" (Nov., 1921), by Langevin. But, in any case, the restricted Relativity principle concerns itself alone with uniform motion; it has nothing to say on the relativity of rotation and acceleration. That will come up in the second part of this volume, and be discussed there.

* "C.R.," 157 (1913), pp. 708 and 1410. "J. de Phys." (5) 4, (1914), p. 177.

CHAPTER III.

So far we have been concerned with the relations which exist between space and time co-ordinates in two frames, S and S′, in uniform relative motion to one another, and between kinematic measures in these frames. For this purpose we have only required the so-called second postulate of Relativity, the invariance of the velocity of light. We now come to considerations where the first postulate makes a direct appearance, the invariance of the expressions of the laws of nature.

We are, in this connection, concerned not with verbal statements of these laws so much as with the equations which are the mathematical expression of these laws. All science of an exact nature professes to deal with measured quantities, and to find uniformities and sequences in these measures. In physics the main object in studying any natural phenomenon is to find the manner in which a measured quantity in one place and at one time is related to its measures at other places and times. To this end any theory which professes to be true must lead to differential equations, the solution of which would presumably determine such relations and determine them correctly, i.e., in such a way that they would agree within the limits of experimental error with all observations in whose precision we have the necessary confidence. We then say that a law is " true to nature." But if we accept the standpoint of Relativity, we subject such laws to a second test. In the differential equations, the variables with regard to which we differentiate the symbols which represent the measurable quantities are the space and time co-ordinates. At once the question of a frame of reference arises, and in relativity no privilege is given to one frame above another. The form of the differential equations must be the same in S as in S′; a solution would yield a function of x, y, z, t, which would fit with the measures of the quantity in S, and exactly the same function of x', y', z', t' should fit with the measures in S′.

It has been pointed out in the Introduction that this test of Relativity would be satisfied by the Newtonian laws of motion

if the relations between the co-ordinates and time in the two frames were embodied in the equations

$$\begin{aligned} x' &= x - ut \\ y' &= y;\ z' = z \\ t' &= t. \end{aligned}$$

But, as was shown, these equations are inconsistent with an equal velocity of light in the two frames; moreover they do not allow the very important and fundamental equations of electromagnetic theory to assume the same form in the frame S as they do in S′ without introducing, as Lorentz did, hypotheses concerning contractions accompanying motion through the ether, and such concepts as "effective co-ordinates" and "local time." On the other hand, if one uses the Lorentz transformation, the electromagnetic equations pass the relativity test, but the Newtonian equations of motion do not. This, at once, raises in an acute form the question of the universal validity of Newton's laws. To some minds such a challenge seems almost blasphemous. In the early days of Maxwell's theory of the electromagnetic field, it would not have been considered for a moment as a possibility; but from the year 1881, when J. J. Thomson first introduced the idea of the "electromagnetic inertia" of a moving charge, theory and practice have combined gradually to accustom men's minds, not to an abandonment, but to a revision and generalisation of the Newtonian laws. In particular, it was perceived that the experiments on high-speed electrons compel physicists to relinquish the notion of invariable mass. It is true that the added electromagnetic mass can be associated, if one likes, with that dragging along of Faraday tubes, of which the magnetic field of a moving charge is conceived to be the outward and visible manifestation. Nevertheless, an attitude of scepticism was more and more observable, and the relativist was really rendering a signal service in bringing the matter to a head, and in offering for consideration a set of equations to which Newton's are an approximation in certain limiting conditions, which satisfy the relativity test, and which are true to nature under conditions for which Newton naturally had no experimental data to go upon.

In the Newtonian scheme, the differential equations for a body are derived from the statement that the force is proportional to and codirectional with, the rate of charge of momentum, which leads to the differential equations

$$d(m\,dx/dt)/dt = F_x,$$

and two similar equations;

or $$m\,d^2x/dt^2 = F_x,$$

and two similar equations.

An essential feature of the Newtonian system is the absolute constancy and invariability of m for the body, so that m has the same value for all positions and speeds of the body. A further feature is the nature of the quantity represented by the symbol appearing on the right-hand side. A great deal of discussion has raged around the nature of force and the possibility of action at a distance as contrasted with transmission of stress through a continuous medium, but for our immediate purpose the symbol **F** stands for a quantity depending on the relative position of the body to other bodies said to be " exerting force " on it; that is, it is a function of the co-ordinates of the other bodies relative to a definite point in this body as origin. At least, that is so in the case of conservative forces. In motion involving dissipation of energy as heat, **F** would also include terms depending on the relative motion of the body and the other bodies referred to.

To sum up, we equate two vector quantities, one involving the second differential coefficient of a radius vector with respect to time, and the other involving no such differential coefficients, or, at most, first differential coefficients; so that, given sufficient information about the position and motion at a given instant (the " initial conditions "), and the manner in which **F** depends on position and motion, we can evolve by solution the manner in which x, y, z depend on t, and obtain the path of the body, idealised, of course, as a particle concentrated at its centre of mass. In fact, t is the independent variable upon which x, y, z depend, and the object of the solution is to express these latter symbols as functions of t.

As already stated, the possibility of retaining an invariable m in the equations as a perfectly general condition has been seriously questioned for some time, even before the enunciation of the Relativity principle; and, in any case, they cannot be made to square with relativity, if one retains the Lorentz transformation equations. For if the above equations are true in the frame S, then it is not true in the frame S′, that

$$m\,d^2x'/dt'^2 = F_x', \text{ etc.}$$

where **F′** is the transformed **F**, the Lorentz equations being employed in the transformation.

From the standpoint of Relativity, we must endeavour to suggest equations of motion which will transform unchanged in form; and, of course, as Newton's equations have passed unchallenged so long in conditions where the material velocities are small compared to that of light, such suggested equations must degenerate towards the Newtonian form as a limit as the velocity approaches zero. Attention should be carefully focussed on this procedure; it is typical in Relativity theory. Relativity does not deny the possible validity of well-attested laws as extremely good approximations in conditions of comparatively slow motion. It gladly welcomes such knowledge as a very necessary, and, indeed, indispensable, help towards the discovery of more general laws which will accord with the principle, and which (it is hoped) will turn out to be true to nature under circumstances not hitherto contemplated. The problem is, in fact, twofold: in the first place, existing knowledge has to be generalised, accordance with Relativity being a test to which all suggested generalisations must conform; in the second, experimental work must be carried out where possible on the validity of the suggestions under broader conditions. We can illustrate the procedure admirably in the present instance.

In the first case, it has been noticed that in the Newtonian equations, t is the *independent* variable. Now, it is a feature of Relativity that the time measured in a given frame, and the space co-ordinates in that frame, are much more closely linked than before. So close is the union, that there is, as stated, a strong impulse in certain quarters to regard time and space as a unity which we split in two for our own immediate convenience, but with no philosophical justification. Be that as it may, it is no longer possible in the differential equations conforming with Relativity to regard t as an independent variable; it must be put on the same footing as the space co-ordinates x, y, z, and regarded as a dependent variable. But dependent on what? It is here that we begin to perceive the extreme importance of the concept of proper time introduced in the previous chapter. As pointed out there, an interval of proper time measured from a definite point-instant to another definite point-instant is the same in all frames; it is invariant. It retains, therefore, sufficient of the absolute nature that was formerly postulated for time, measured anywhere, to render it suitable as the independent variable in our differential equations. Further, in using the convention of splitting vector displacement into three Cartesian components, we have hitherto

employed groups of three equations. Now, we must be prepared for groups of four, since differential equations must contain the coefficients of x, y, z and t with respect to τ.

Let us try

$$\left.\begin{aligned} d(\mu dx/d\tau)/d\tau &= P_x \\ d(\mu dy/d\tau)/d\tau &= P_y \\ d(\mu dz/d\tau)/d\tau &= P_z \\ d(\mu dt/d\tau)/d\tau &= P_t \end{aligned}\right\} \quad . \quad . \quad . \quad (1)$$

Concerning these, a few remarks are necessary before proceeding.

1. We must endeavour to keep out of mind at the moment terms like *mass* and *force* as used in the Newtonian theory. So as to give no unconscious bias in that direction, the symbols m and $\mathbf{F}$ are not employed at this stage. The terms will be reintroduced later, but their definition as used in relativity arises directly out of the suggested equations, and they are not regarded as entities having a meaning apart from the equations.

2. There are four equations.

3. The proper time τ is the independent invariable.

4. μ is regarded as an invariant quantity for the particle whose path in a given frame is implicit in these equations. It may differ for different particles. But for a given particle, it is at a given point-instant a quantity measured alike by all observers; if it is μ in S, μ' in S$'$, μ'' in S$''$, and so on, then $\mu = \mu' = \mu'' =$ etc., so that the use of accents in connection with it is quite unnecessary. This assumption *corresponds to* the assumption of constant mass in Newtonian dynamics, but must, on no account, be confused with it. It is a much more general assumption, as will be seen later, and for that reason we shall endeavour to keep the issue clear by giving it no name at present, least of all "mass." Further, it may be as well to point out that "invariance" must not be confused with "invariability" or "constancy." At a given point-instant for the particle, all observers use the same value of μ; that assumes invariance. At another given point-instant all observers will again use the same value of μ—invariance once more; but there is no assumption at present that the first μ and the second μ shall be equal—that would be assuming constancy or invariability. To put it another way, it is quite possible that μ might be a function of x, y, z, t—at least, that possibility is not ruled out in the assumption of *invariance*. If the particle is at the point-instant whose co-ordinates are (x, y, z, t) in S, and

(x', y', z', t') in S′ (connected by the Lorentz equations), then

$$\mu(x, y, z, t) = \mu(x', y', z', t').$$

But if (x_1, y_1, z_1, t_1), and (x_2, y_2, z_2, t_2), are the space-time coordinates of two point-instants of the particle in one frame S, then there is no assumption so far that

$$\mu(x_1, y_1, z_1, t_1) = \mu(x_2, y_2, z_2, t_2).$$

The absence of any assumption as to the constancy of μ is implicit in the form we have given the left-hand side of the equations (1), where $d(\mu dx/d\tau)/d\tau$ is written, and not $\mu d^2x/d\tau^2$.

5. P_x, P_y, P_z, P_t are assumed to be functions of the coordinates and velocity of the particle; but again, we refrain from introducing the term "force" at this stage.

6. The first three equations are clearly sufficiently similar in form to the Newtonian triad to pass for a formal generalisation of them, into which we can reasonably hope to read a definite meaning (true or otherwise), but the fourth looks decidedly peculiar at a first glance. It will turn out, however, to be of first-rate importance, an old friend in disguise, in fact; but for the moment we shall fix our attention on the first three.

The first thing to satisfy ourselves about is the test of transformation. If the equations do not satisfy the first postulate of Relativity, they are of no interest to us as relativists.

In considering the transformation from S to S′, we have by equations (4) and (5) of the previous chapter (remembering the invariance of μ),

$$\begin{aligned} d(\mu dx'/d\tau)/d\tau &= a(P_x - uP_t) \\ d(\mu dy'/d\tau)/d\tau &= P_y \\ d(\mu dz'/d\tau)/d\tau &= P_z \\ d(\mu dt'/d\tau)/d\tau &= a(P_t - uP_x). \end{aligned}$$

Hence for the invariance of the equations it is necessary that the measures made in S′ of the quantities involved in the right-hand side of the equations (1), viz., P_x', P_y', P_z', P_t', be related to the measures in S by the equations

$$\left.\begin{aligned} P_x' &= a(P_x - uP_t) \\ P_y' &= P_y;\ P_z' = P_z \\ P_t' &= a(P_t - uP_x) \end{aligned}\right\} \quad . \quad . \quad . \quad (2)$$

and of necessity also by the equations

$$\left.\begin{aligned} P_x &= a(P_x' - u'P_t') \\ P_y &= P_y';\ P_z = P_z' \\ P_t &= a(P_t' - u'P_x') \end{aligned}\right\} \quad . \quad . \quad . \quad (2\text{A})$$

In short, the quantities P_x, P_y, P_z, P_t must be cogredient with x, y, z, t. This is a condition we shall have to bear in mind and return to when we elicit, as we shall do shortly, some more definite information as to the nature of these quantities.

Passing on to the question of "truth to nature," we first find the form to which these equations degenerate if the velocity of the particle in a given frame S is taken to be comparatively small, i.e., if v is a vanishingly small fraction.

Remembering that $d/d\tau = \beta d/dt$, we see that

$$d(\mu\beta dx/dt)/dt = P_x/\beta \quad . \quad . \quad . \quad (3)$$

and two similar equations. Now as v approaches the value zero, β approaches unity, and $d\beta/dt$ (which $= d(1 - v^2)^{-\frac{1}{2}}/dt = \beta^3 v dv/dt$) approaches zero.

Hence the equations (3) degenerate to the form

$$d(\mu dx/dt)/dt = P_x$$

and two similar equations, and these are the Newtonian equations provided we interpret μ as the "mass" of the particle, and **P** as the "force exerted on it." So our suggested equations satisfy Relativity if equations (2) are satisfied, and they contain the Newtonian equations as a limiting form. It remains to see if they are in agreement with any known experimental results concerning bodies travelling with comparatively high speeds. Recent observations on the deviation of electrons by electric and magnetic fields are decisively in favour of the view that if the force acting on the electron is calculated in terms of the electric and magnetic intensities of the applied fields in the manner suggested by electromagnetic theory, then the measure of the electron's inertia (i.e., the factor which multiplies the velocity in the calculation of momentum) is not a constant, but increases very markedly as the velocity increases, and the quantitative results agree, even up to speeds as high as ·8 of the speed of light, with the assumption that the inertia varies inversely as $(1 - v^2)^{\frac{1}{2}}$, v being the ratio of the electron's speed to that of light.

Now it is clear that this conclusion is contained in equations (3), provided we interpret the expressions P_x/β, P_y/β, P_z/β as the "force." Suppose, then, using the symbols **F** and F_x, F_y, F_z for "force" and its components, we rewrite (3) as

$$d(mdx/dt)/dt = F_x \quad . \quad . \quad . \quad (3\text{A})$$

and two similar equations where

$$m = \mu\beta = \mu/(1 - v^2)^{\frac{1}{2}}.$$

We may refer to m as the "mass" of the particle, but there is no constancy about it. Still, there is nothing in these equations contrary to any known facts. Assumption of a constant μ will cover all known results. It is customary to refer to this quantity as the "rest-mass," for it is the limit of the "mass" as v approaches zero.* What is most striking is the ease with which the Relativity principle, in the character of a test to which equations of motion must conform, arrives at a result as regards inertia *in general*, which had already been worked out from general electromagnetic theory by Lorentz (with the aid of special hypotheses) *for the electron*, and which, as stated, has been amply verified by experience. It must be borne in mind that we are not dealing here with mass as a "quantity of matter" or any other vague concept. The word is used in the sense of "inertial mass," i.e., a factor in "momentum," which, in its turn, is the quantity whose rate of change per unit of time measured in a given frame is equal to the "force," which is another quantity supposed to be calculable by definite and experimentally-justifiable rules. With many of these rules we are all familiar, e.g., the well-known inverse square laws for purely gravitational, electrostatic, and magnetostatic observations. But in this connection a word of warning must be uttered. A few paragraphs back, attention was drawn to the fact that the quantities P_x, P_y, P_z, P_t must be cogredient with x, y, z, t if our suggested equations of motion are to lie within the bounds of the Relativity hypothesis. Consequently, whatever be the rules by which we calculate forces, it follows that βF_x, βF_y, βF_z, and P_t must be cogredient with the space-time co-ordinates. "But," one may naturally ask, "what is P_t?" In that thorough-going form of the analysis employed in Relativity problems, which makes full use of the idea of a four-dimensional space-time, one speaks of time component with the same justification as space-components. But at this juncture we had better avail ourselves of more familiar conceptions. If we revert to equation (3) of the second chapter, we see that

$$\dot{t}^2 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2 = 1,$$

and hence $$\dot{t}\ddot{t} - \dot{x}\ddot{x} - \dot{y}\ddot{y} - \dot{z}\ddot{z} = 0$$

or $$\ddot{t} = \ddot{x}v_x + \ddot{y}v_y + \ddot{z}v_z, \quad . \quad . \quad . \quad (4)$$

which is really only another way of writing equation (6) of that chapter.

* "Proper" mass may also be used.

So if we assume that μ is a constant for a given particle as well as an invariant, it follows from (4) that

$$d(\mu\dot{t})/d\tau = \Sigma v_x d(\mu\dot{x})/d\tau,$$

and hence by (1)

$$\begin{aligned} P_t &= \Sigma(v_x P_x) \\ &= \beta\Sigma(v_x F_x) \\ &= \beta(\mathrm{v}\,.\,\mathbf{F}), \qquad (5) \end{aligned}$$

using the usual vector symbol for the geometric product of velocity and force. As a consequence of this result, it follows that in order that the suggested equations of motion may survive the relativity test, any suggested law of force must permit the quantities

$$\beta F_x,\ \beta F_y,\ \beta F_z,\ \beta(\mathrm{v}\,.\,\mathbf{F})$$

to be cogredient with the space-time co-ordinates. In short, if the law when employed by observers in S gives F_x, F_y, F_z as the components of the force on a particle at a given point-instant, and the same law as employed by observers in S′ gives F_x', F_y', F_z' as the components at the same point-instant, then

$$\left.\begin{aligned} \beta' F_x' &= \alpha[\beta F_x - u\beta(\mathrm{v}\,.\,\mathbf{F})] \\ \beta F'_y{}' &= \beta F_y\,;\ \beta' F_z' = \beta F_z \\ \beta'(\mathrm{v}'\,.\,\mathbf{F}') &= \alpha[\beta(\mathrm{v}\,.\,\mathbf{F}) - u\beta F_x] \end{aligned}\right\} \qquad (6)$$

It will appear later that when subjected to this test, the law for calculating the force on a charged body moving in an electromagnetic field is satisfactory; but the well-known Newtonian inverse square law for gravitational force is not. But these matters will be deferred to the next chapter. We will return at the moment to the discussion of the broadened conception of mass, and we now find ourselves in a position to interpret the fourth member of equations (1), whose peculiarity we noted at the outset. We can now write it

$$d(\mu\beta)/dt = P_t/\beta$$

or

$$dm/dt = (\mathrm{v}\,.\,\mathbf{F}) \qquad (7)$$

It was stated that this equation would turn out to be an old friend in disguise. It is, in fact, the equation of energy. For multiplying each side of (7) by an element of time δt (for the frame S), we get

$$\delta m = \Sigma F_x \delta x.$$

The right-hand side is, if interpreted on the usual lines, the work of the force on the particle while it is displaced along an

element of path whose components are $(\delta x, \delta y, \delta z)$. C..
δm is the increment of the particle's *kinetic energy* dur..
displacement, and thus the kinetic energy of the particle ..
be the integral of dm from the condition of rest. But at res.
the value of m is μ. Hence the kinetic energy of the particle, when its velocity is v, is equal to $m - \mu$ or $\mu(\beta - 1)$.

As one may reasonably expect, this extremely important result degenerates to the usual expression for small velocities, for

$$\begin{aligned}\mu(\beta - 1) &= \mu[(1 - v^2)^{-\frac{1}{2}} - 1] \\ &= \mu[\tfrac{1}{2}v^2 + \tfrac{3}{8}v^4 + \ldots],\end{aligned}$$

which approaches $\frac{1}{2}\mu v^2$ as v approaches zero. But, of course, the really striking feature of this result is the establishment of the identity of kinetic energy with the increase of mass due to the motion. It requires some mental effort on the part of those unacquainted with recent electromagnetic theory to take to such a notion kindly. But at the outset the warning was uttered that we were to disabuse our minds of all preconceived notions of mass, momentum, and force; that these words were to be accepted as convenient names for certain terms in the suggested equations of motion, which formally make their appearance in the same character as do terms in the Newtonian equations bearing the same names. To those, however, who know some electromagnetic theory, this apparent identity of mass and energy will not come as a shock. As has been mentioned, almost forty years ago J. J. Thomson showed that a charged body has its inertial mass increased by the fact of its electrification, and the amount of this increase is proportional to e^2/a where e is the charge and a a linear dimension of the body (its radius in the case of a sphere). He further showed that this "electromagnetic mass" increased with the speed of the charged body. Later it was found that the assumption that the whole mass of an electron could be accounted for in this electromagnetic manner was perfectly justified by experiment; and, finally, Lorentz showed by a special hypothesis concerning the deformability of the electron, that the "rest-mass" of an electron was proportional to e^2/a, and the mass at speed v was proportional to $\beta e^2/a$, the speed being interpreted as speed through the ether. Now the point to be emphasised in this connection is that e^2/a has the dimensions of energy in electrical theory. Therefore, so far from arriving at a nonsensical result, Relativity has deduced as a general proposition a conclusion which had already been suggested by a special

physics in a special case, with the aid of certain assumptions. Moreover, if we are to identify energy of motion with increase of mass due to motion, it is but a natural process to identify the entire mass of a body as a measure of energy also, so that the "rest-mass" would be but another name for an amount of energy "intrinsic" to the body, and existing in it, independent of its motion. There is nothing to oppose such an assumption, and a good deal to support it. We have seen it justified in the case of the electron. As further evidence, we have only to mention the well-known fact that most theories as to atomic structure which are being propounded and developed to-day, assign practically the entire mass of an atom to its positive nucleus, and do so by postulating an extremely small nucleus, so that the electromagnetic measure of the mass may produce a sufficiently large result by reason of the small linear quantity occurring in the denominator of the expression e^2/a. There is still a more striking illustration of the assent which is being given to the belief that energy and mass are identical. In attempting to account for the periodic properties of elements, suggestions have been put forward that all atomic nuclei are built up of hydrogen nuclei or "protons," with attendant electrons. This would apparently imply that all atomic weights could be obtained by adding simple multiples of the weight of a proton and of the weight of an electron. Now this is not so, but the discrepancies between the actual atomic masses and those calculated on this theory of atomic structure are accounted for by what is called "nuclear packing." In simple words, it is assumed that if these ultimate electrified particles have their mutual distances of separation altered so that their ordinary electrostatic *potential* energy is altered, then also is their total inertial mass as a group altered, and it is not justifiable to regard this total mass as a mere arithmetical sum of the separate masses of the particles when isolated, but as a quantity which varies with change of internal structure in the group, just as the energy of the group varies. It is, indeed, a sign of the power of the Relativity principle that it unifies in a direct and simple way so much apparently unrelated material in our physical and chemical schemes. It is a far cry from the Michelson-Morley experiment to the periodic table of the elements, yet Relativity discloses an unsuspected relation.

We have hitherto made no use of a possible variability of μ, the rest-mass or intrinsic-energy (whichever name we choose to give it), although we have hinted at such a possibility. (In this

connection it may not be out of place to reiterate
against confusing "constancy" with "invariance.")
have allowed for it, nevertheless, in our equations by wi
$d(\mu dx/d\tau)/d\tau$, etc., at the outset, and not $\mu d^2x/d\tau^2$, etc. Also, in (7) we see that a finite dm/dt might be the result of an alteration in μ as well as an alteration in v and β; so that it would be theoretically possible for a force to be accompanied not by an acceleration of the body's motion, but by an alteration in its intrinsic energy. To illustrate this possibility, let us consider a case which was once proposed as an example in which Relativity led to a contradictory result. A body is at rest in the frame S, and is radiating energy. Obviously, there is no ground for believing that it would begin to move in S, if isolated from the action of other bodies. Hence, in the frame S′, it should move with a uniform velocity, viz., the relative velocity of the frames. But on electromagnetic theory such a body would, if in motion, be subject to a resistance due to its own radiation, and therefore should be retarded in S′. The reconciliation lies in the fact that if we offer to discuss this apparent contradiction we must, in order to justify the Relativity result, use Relativity mechanics, which does not deny the conclusion that there is a "retarding" force as obtained by electromagnetic theory, but denies the necessity of a retardation on that account. Decrease of momentum there must be, but the decrease is due to the diminution of the inertia factor and not of the velocity factor; for as the body is radiating, its intrinsic energy grows less, i.e., its rest-mass is decreasing.

We should have to admit in a similar fashion that if a body were rising in temperature, it would require a force to maintain it in uniform motion in a given frame, on account of its increasing intrinsic energy. If any astonished person inquires for the "mass of 1 calorie of heat," he will not be perpetrating a joke, but asking a perfectly reasonable question, and one to which an answer is possible. Let it be recalled what fundamental units we are using; they are 1 centimetre for length, and $1/c$ second for time. Hence our unit of velocity is c cms. per sec., and our unit of acceleration is c^2 cms. per sec. per sec. So if we wish to retain the erg as our unit of energy, and the dyne as our unit of force, we must use $1/c^2$ gram as our unit of mass; for a body with that rest-mass, and moving off with an initial acceleration equal to our unit of acceleration, would be subject to a force of 1 dyne. In brief, just as mass and energy have become identified, we find also that 1 erg is equal to $1/c^2$ gram,

...dislikes that way of putting it, we can say that ...ergy of a body, intrinsic or kinetic, increases by c^2, ...9 $\times 10^{20}$ ergs, its inertial mass increases by 1 gram.

Now suppose 1 calorie of heat is absorbed by a body without performance of external work by it, the intrinsic energy has increased by $4{\cdot}2 \times 10^7$ ergs, and therefore the body's mass is greater by $(4{\cdot}2 \times 10^7)/(9 \times 10^{20})$, or $4{\cdot}7 \times 10^{-14}$ gram. Such a numerical result does not hold out much hope of testing this conclusion by experiment, but it is a justifiable deduction from Relativity.

The term, "adiabatic" motion, is a convenient one to apply to motion in which the intrinsic energy or rest-mass of a body remains unaltered.

Questions which cannot be more than referred to here will naturally arise in the inquiring mind. We have grown accustomed since the days of Maxwell and Faraday to conceive of electrostatic and magnetic energy as localised in space, "in the ether," in contrast to the view of mathematicians of the eighteenth and early nineteenth centuries, who concentrated their attention on the "charges" of electricity and magnetism in matter. No such diffusion was more than hinted at in the case of gravitational energy, but undoubtedly all people regarded the energy of strain (a form of potential energy) as localised in the bent, stretched or twisted material. As a result, anyone may now justly ask if we are to diffuse the mass accompanying potential energy in precisely the same way. The answer must be in the affirmative. Indeed, we have already as common features in our text-books of advanced physics such phrases as radiation-pressure, the electromagnetic mass and momentum of a finite beam of radiant energy, etc. What the Relativity theory does is to make such a diffusion a natural feature of its mechanics from the beginning, not a conclusion which is reached from advanced considerations in a special branch of physics. If a body is decreasing in speed, it is losing energy and mass. The one loss cannot take place without the other. This energy and mass may presently appear in other bodies; both must appear together (not merely energy alone), but in the meanwhile, since nothing is propagated with a speed greater than that of light, they must have been localised in a definite volume between the bodies, i.e., as we say, "in space," at each definite instant. There is no escaping the conclusion. No doubt it is going to complicate "Advanced Dynamics" for our honours students in the future! For one thing, it will not allow us to localise all mass in a physical

system quite so definitely as before. The mass o liquid body always seems a very real and tangible t which we have clear muscular perception ; the mass of a gas substance is only too evident in a gale, and even without tha there is the balance to show us the difference between a flask before and after evacuation ; but there is something very shadowy about the mass resident " in space." It is true that in practice the amount within any finite volume is in general excessively minute compared with that possessed by an equal-sized piece of " matter," but the fact remains that " matter " is not going to be quite so easily defined as formerly. If we have to ascribe mass to the somewhat ghostly haunts of energy designated by the phrases, " the gravitational " and " electro-magnetic field," the distinction between energy and matter becomes one of mere practical convenience.

But further discussion on these topics is out of place here. For those who can follow the mathematics, the points raised will be treated by suitable analysis at a later stage, and it will be shown that we still preserve the conservation laws in Relativity. Conservation of mass and conservation of energy, of course, merge into one law ; but both it and conservation of momentum require us to diffuse mass and momentum in the manner indicated, a conceptual process quite foreign to Newtonian mechanics.

Far more important for us at this point is the realisation that mass and force are relative concepts. Equations (6) show us that with any laws of force measurement which can be made to accord with Relativity, a force will have a different value in one frame of reference to that in another. Equations (3A) make it clear that the mass of a body varies for different observers, actually reaching the value infinity for an observer if it could attain the velocity of light with respect to him. This is the dynamical reason for the view that the speed of a body cannot surpass that of light. The material of this chapter can be readily summarised in the statement that we have added two more tetrads to those groups of four physical quantities which we say are cogredient with the space-time co-ordinates. They are the " force-activity " tetrad,

$$\beta F_x,\ \beta F_y,\ \beta F_z,\ \beta(\mathbf{v}\ .\ \mathbf{F}),$$

and the " momentum-energy " tetrad,

$$mv_x,\ mv_y,\ mv_z,\ m,^*$$

* That these four are cogredient follows from the cogrediency of βv_x, βv_y, βv_z, β demonstrated in Chapter II., and the postulated invariance of μ.

...es of variation of the latter with respect to *proper* ... individually equal to the former.

It is necessary to be on one's guard against a not unnatural misinterpretation of the Relativity laws of motion. In Newtonian mechanics, rate of charge of momentum and product of mass and acceleration are in general interchangeable expressions, and either can be equated to the force. That is due to the assumed constancy of the mass. It is quite otherwise in the new mechanics, e.g.,

$$\left.\begin{aligned} F_x &= d(mv_x)/dt \\ &= mdv_x/dt + v_x dm/dt \\ &= ma_x + \beta^2 mvv_x dv/dt \\ \text{Similarly,}\quad F_y &= ma_y + \beta^2 mvv_y dv/dt \\ F_z &= ma_z + \beta^2 mvv_z dv/dt \end{aligned}\right\} \quad . \quad . \quad . \quad (8)$$

Hence we see that product of mass and acceleration is only one part of the force exerted on a body, that in addition to a component of force equal to the mass-acceleration product, and in the direction of acceleration, there is a further component in the direction of the velocity, and equal to

$$\beta^2 mv^2 dv/dt.$$

This latter component vanishes for constant speed (magnitude of velocity). This is the case when the acceleration is normal to the velocity in direction. In this case the equations reduce to the form

$$F_x = ma_x$$

and two similar equations, the force being also normal to the path of the particle. Another special case arises when the direction of motion is unchanged. Let this direction be the axis of x; then, dropping suffixes for the moment as unnecessary,

$$\begin{aligned} F &= ma + \beta^2 mv^2 a \\ &= ma/(1 - v^2) \\ &= \mu a/(1 - v^2)^{\frac{3}{2}}. \end{aligned}$$

Thus, in the special case of a "transversal" force, the quotient of force by acceleration is the quantity m or $\mu/(1 - v^2)^{\frac{1}{2}}$, and in the case of a "longitudinal" force the quotient is $m/(1 - v^2)$ or $\mu/(1 - v^2)^{\frac{3}{2}}$. On this account these quantities are sometimes called the "transversal" and "longitudinal" masses respectively. It should be remarked, in passing, that the experiments on the increased mass of high-speed electrons have all been of the "transversal" type. Since

$$vdv/dt = \Sigma v_x dv_x/dt = \Sigma v_x a_x,$$

it is not difficult to show that equations (8) reduce to

$$\left.\begin{aligned} F_x &= \beta^3\mu\{(1 - v_y^2 - v_z^2)a_x + v_xv_ya_y + v_xv_za_z\} \\ F_y &= \beta^3\mu\{v_yv_xa_x + (1 - v_z^2 - v_x^2)a_y + v_yv_za_z\} \\ F_z &= \beta^3\mu\{v_zv_xa_x + v_zv_ya_z + (1 - v_x^2 - v_y^2)a_z\} \end{aligned}\right\} \quad . \quad (9)$$

These rather complicated looking equations can be shown to embody the result of blending the two special cases referred to. For, suppose at a given instant we choose the axis of x to be the tangent to the particle's path, the axis of y to be its principal normal, and the axis of z to be normal to its osculating plane, then

$$v_y = v_z = 0$$

and $$a_z = 0.$$

Hence, writing v, a_t and a_n, for v_x, a_x and a_y respectively, we obtain

$$\left.\begin{aligned} F_t &= \beta^3\mu a_t \text{ (tangential components)} \\ F_n &= \beta^3\mu(1 - v^2)a_n \text{ (normal components)} \\ &= \beta\mu a_n \end{aligned}\right\} \quad . \quad (10)$$

and the remaining component of force is zero. Hence, if we resolve along tangent and principal normal, the tangential or "longitudinal" component of force is equal to the product of the corresponding component of acceleration and the "longitudinal" mass, while the normal or transversal component of force involves the "transversal" mass as the factor of the normal acceleration. This ambiguity about mass, however, only arises in connection with the mass-acceleration rule for force; there is no doubt about mass when we use the form of the law involving rate of change of momentum, and, as a general rule, it is in that sense (a factor of momentum) in which it is used, being then also a measure of a body's total energy.

As a direct deduction from equations (8), (9), or (10), it appears that uniform acceleration does not imply a uniform field of force, another break with the familiar views of the older mechanics. If, for instance, we consider the comparatively simple case of a body moving in one direction under a constant force, we put $F_y = F_z = 0$, $F_x =$ constant and $v_y = v_z = 0$. Then, by equations (9),

$$F_x = \beta^3\mu a_x.$$

Hence the acceleration in the frame S is not constant, but varies as $(1 - v^2)^{\frac{3}{2}}$. Referring to equation (10), of the second chapter, we see that this is a case of uniform "*rest-acceleration.*"

the motion could not be one of uniform acceleration, the speed increases the mass increases without limit as the velocity of light is approached, necessitating a proportional and likewise unlimited increase in the force to maintain a constant acceleration. Another way of arriving at this result is to consider at any instant a frame S′ in which the body is momentarily at rest, the "rest-frame." Then

$$u = v_x,\ \alpha = \beta,\ v' = 0,\ \beta' = 1.$$

So
$$\beta' F_x' = \alpha\{\beta F_x - u\beta(\mathbf{v}.\,\mathbf{F})\}$$
$$= \alpha\beta(F_x - uv_x F_x),$$

i.e.,
$$F_x' = \beta^2(1 - v_x^2)F_x = F_x.$$

Hence the force in the frame which is at any instant the rest-frame is always the same as that in the specially chosen frame S, and so is constant likewise. Since

$$F_x' = \mu a_x'$$

where a_x' is the rest acceleration, it follows that a_x' is constant.

But even this element of simplicity disappears when we have curvilinear motion under a constant force. In classical mechanics we have parabolic motion obtained by compounding uniformly accelerated motion in one direction with uniform motion in a perpendicular direction, and it would not be unnatural for anyone to jump to the conclusion that in the new mechanics a trajectory under a constant force would be a hyperbola, obtained by combining the rectilinear motion having a uniform rest acceleration, and given by the equation

$$x^2 - t^2 = (1/g)^2$$

(see equation (12), Chapter II.), with the rectilinear uniform motion,

$$y = kt.$$

The fallacy in this procedure, however, will be evident on considering the increase in mass; for since the mass increases with the speed, it will require a y-component of force to maintain a constant y-component of velocity; and as the speed at a point of a hyperbolic trajectory is greater than the speed at the corresponding point of the rectilinear path with uniform rest-acceleration, it will also require an increasing x-component of force to maintain an x-component of acceleration, varying as $(1 - v_x^2)^{\frac{3}{2}}$, so as to keep the x co-ordinate still connected with the time by the equation $x^2 - t^2 =$ constant. This is very manifest on taking equations (9) for two dimensions, with a

x, y, z, t in general, and by the transformation equations can also be expressed as functions of x', y', z', t'. So making use of equations (6), we see that the equations (2) can be written:

$$\begin{aligned}
&a(\partial E_x/\partial t' - u\partial E_x/\partial x') + \rho v_x = \partial H_z/\partial y' - \partial H_y/\partial z' \\
&a(\partial E_y/\partial t' - u\partial E_y/\partial x') + \rho v_y = \partial H_x/\partial z' - a(\partial H_z/\partial x' - u\partial H_z/\partial t') \\
&a(\partial E_z/\partial t' - u\partial E_z/\partial x') + \rho v_z = a(\partial H_y/\partial x' - u\partial H_y/\partial t') - \partial H_x/\partial y'
\end{aligned}$$

The second and third of these happen to be more easily manipulated than the first, so we shall deal with them prior to the first. The second, for instance, requires little rearrangement to give

$$\partial\{a(E_y - uH_z)\}/\partial t' + \rho v_y = \partial H_x/\partial z' - \partial\{a(H_z - uE_y)\}/\partial x',$$

and the third yields

$$\partial\{a(E_z + uH_y)\}/\partial t' + \rho v_z = \partial\{a(H_y + uE_z)\}/\partial x' - \partial H_x/\partial y'.$$

So if we put

$$\begin{aligned}
E_y' &= a(E_y - uH_z) \\
E_z' &= a(E_z + uH_y) \\
H_x' &= H_x \\
H_y' &= a(H_y + uE_z) \\
H_z' &= a(H_z - uE_y) \\
\rho' v_y' &= \rho v_y \\
\rho' v_z' &= \rho v_z
\end{aligned}$$

we could write these two in the form

$$\begin{aligned}
\partial E_y'/\partial t' + \rho' v_y' &= \partial H_x'/\partial z' - \partial H_z'/\partial x' \\
\partial E_z'/\partial t' + \rho' v_z' &= \partial H_y'/\partial x' - \partial H_x'/\partial y'.
\end{aligned}$$

These relations also give us the hint as how to proceed with the first equation above, for we find that

$$\begin{aligned}
&\partial\{a(H_z - uE_y)\}/\partial y' - \partial\{a(H_y + uE_z)\}/\partial z' \\
&= a(\partial H_z/\partial y' - \partial H_y/\partial z') - au(\partial E_y/\partial y' + \partial E_z/\partial z') \\
&= a(\partial H_z/\partial y - \partial H_y/\partial z) - au(\partial E_y/\partial y + \partial E_z/\partial z) \\
&= a(\partial E_x/\partial t + \rho v_x) - au(\rho - \partial E_x/\partial x) \qquad \text{by (2) and (4)} \\
&= a^2(\partial E_x/\partial t' - u\partial E_x/\partial x') + a\rho(v_x - u) + a^2u(\partial E_x/\partial x' - u\partial E_x/\partial t') \quad \text{by (6)} \\
&= a^2(1 - u^2)\partial E_x/\partial t' + a\rho(v_x - u) \\
&= \partial E_x/\partial t' + a\rho(v_x - u).
\end{aligned}$$

Hence, if in addition to the above symbols, we further introduce

$$\begin{aligned}
E_x' &= E_x \\
\rho' v_x' &= a(\rho v_x - u\rho),
\end{aligned}$$

we see that the first equation becomes

$$\partial E_x'/\partial t' + \rho' v_x' = \partial H_z'/\partial y' - \partial H_y'/\partial z'.$$

It is now an easy matter to show that with the same definitions of the accented symbols we can obtain the following from equations (3):

$$-\partial H_x'/\partial t' = \partial E_z'/\partial y' - \partial E_y'/\partial z'$$

and two similar equations.

Further, we obtain from (4) and the definitions of the accented in terms of the unaccented symbols:

$$\begin{aligned}
&\partial E_x'/\partial x' + \partial E_y'/\partial y' + \partial E_z'/\partial z' \\
&= a(\partial E_x'/\partial x - u'\partial E_x'/\partial t + \partial E_y'/\partial y + \partial E_z'/\partial z \\
&= a\partial E_x/\partial x + au\partial E_x/\partial t + \partial\{a(E_y - uH_z)\}/\partial y + \partial\{a(E_z + uH_y)\}/\partial z \\
&= a(\partial E_x/\partial x + \partial E_y/\partial y + \partial E_z/\partial z) - au(\partial H_z/\partial y - \partial H_y/\partial z - \partial E_x/\partial t) \\
&= a(\rho - u\rho v_x),
\end{aligned}$$

which we may write as ρ', adding another and final member to our accented symbols.

It is also easy to show that equation (5) leads directly to

$$\partial H_x'/\partial x' + \partial H_y'/\partial y' + \partial H_z'/\partial z' = 0.$$

Having gone through the gamut, we now can say that the differential equations with respect to x', y', z', t', which relate the quantities E_x', E_y', E_z', H_x', H_y', H_z', $\rho' v_x'$, $\rho' v_y'$, $\rho' v_z'$, ρ' one with another, have precisely the same form as those with respect to x, y, z, t, which relate E_x, E_y, E_z, H_x, H_y, H_z, ρv_x, ρv_y, ρv_z, ρ with each other if

$$E_x' = E_x,\ E_y' = a(E_y - uH_z),\ E_z' = a(E_z + uH_y) \qquad (7)$$

$$H_x' = H_x,\ H_y' = a(H_y + uE_z),\ H_z' = a(H_z - uE_y) \qquad (8)$$

$$\left.\begin{aligned}
\rho' v_x' &= a(\rho v_x - u\rho) \\
\rho' v_y' &= \rho v_y;\ \rho' v_z' = \rho v_z \\
\rho' &= a(\rho - u\rho v_x)
\end{aligned}\right\} \quad . \quad . \quad . \quad (9)$$

These equations supply the answer to the inquiry: "Are the equations of the electromagnetic field invariant?" They are if the measures of the various electromagnetic quantities in S are related to the measures in S′ by equations (7), (8), (9). The notion that electric or magnetic intensities at a given point would be measured differently by different observers was not novel when Einstein first demonstrated the result given above. For example, it would have been naturally urged that a charged body at rest in the ether experiences a force $e\mathbf{E}$ when

at rest in the ether, whether a magnetic field exists or not, but if it moves through the ether with a velocity $\mathbf{u}$, the force on it would be the vectorial sum of $e\mathbf{E}$ and $e[\mathbf{u} \,.\, \mathbf{H}]$ if there were a magnetic field; and an observer moving with the body would, on dividing the force on it by the electric charge, find the electric intensity (as he would call it) to be $\mathbf{E} + [\mathbf{u} \,.\, \mathbf{H}]$; i.e. (with our usual choice of axes), an intensity with components,

$$E_x,\ E_y - uH_z,\ E_z + uH_y.$$

Similarly, by considering a magnet pole of given strength at rest in the ether and then in motion, we should arrive at the result that an observer in motion through the ether would obtain as the magnetic intensity $\mathbf{H} - [\mathbf{u} \,.\, \mathbf{E}]$, or with our axes,

$$H_x,\ H_y + uE_z,\ H_z - uE_y.$$

These would be the results derived from older views untinged by any thought of Relativity, and they, at all events, show the dependence of the electromagnetic vectors on motion. So strong, however, was the impulse towards making the field equations as alike as possible for all observers, no matter what their motion through the ether might be considered to be, that these values for the moving observer were not regarded with any great favour; for if these are combined with the ordinary Newtonian space-time transformations,

$$\begin{aligned} x' &= x - ut \\ y' &= y\,;\ z' = z \\ t' &= t, \end{aligned}$$

the transformed equations of the field lose all the simplicity and symmetry of the original set as written for a frame at rest in the ether. Lorentz, employing his own transformation for space-time co-ordinates, arrived at a set of relations between the accented and unaccented symbols, which almost gave the same form to the field equations for the observer moving through the ether as for him who is at rest in it. It was Einstein who pointed out that if complete invariance be required equations (7), (8), (9) will suffice. But if they are examined it appears that there exists the same reciprocity as we discovered in the case of the co-ordinate transformation. Thus we find that

$$\begin{aligned} &E_x = E_x',\ E_y = a(E_y' - u'H_z')\,;\ E_z = a(E_z' + u'H_y') \\ &H_x = H_x',\ H_y = a(H_y' + u'E_z')\,;\ H_z = a(H_z' - u'E_y') \\ &\rho v_x = a(\rho' v_x' - u'\rho') \\ &\rho v_y = \rho' v_y'\,;\ \rho v_z = \rho' v_z' \\ &\ \ \rho = a(\rho' - u'\rho' v_x'). \end{aligned}$$

In other words, these relations supply no criterion as to whether the S or S′ frame is the one at rest in the ether, or for that matter, that either of them is so "fixed." Once more, motion through the ether has become irrelevant; it serves no useful purpose if complete invariance of the equations be demanded—at all events, so far as uniform motion is concerned.

It will be observed that the Einstein relations differ from those developed on earlier views by the inclusion of the factor α, which, of course, exceeds unity by excessively small amounts for usual velocities of relative motion between matter and matter.

Equations (9) lead to a very important result. It will be observed that these constitute an ordinary Lorentz transformation, and show that ρv_x, ρv_y, ρv_z, ρ are cogredient with the space-time co-ordinates. But we are already aware that βv_x, βv_y, βv_z, β are cogredient (where $\beta = (1 - v^2)^{-\frac{1}{2}}$). It follows at once that

$$\rho/\beta = \rho'/\beta',$$

or the quantity ρ/β is invariant for a given body, and will, therefore, have as its invariant value the density of the charge on the body as estimated by an observer to whom it is at rest; for in his frame $v = 0$, and therefore $\beta = 1$.

Now, if τ be the volume of the body in the frame S, and τ' its volume in S′, we know from Chapter I. that $\beta\tau = \beta'\tau'$, for each of these is the volume of the body in its momentarily proper frame, i.e., as measured by the observer to whom it is at rest. Hence

$$\rho\tau = \rho'\tau'.$$

So the quantity of charge on the body is the same to all observers. This result is of the greatest importance in the Relativity theory It is referred to as the invariance of electric charge on a body. All observers measure the charge to be the same. For those to whom the body is in motion, the diminished volume of the body is just compensated by the increased density of its charge.

This result must not be confused with the well-known result known as the conservation of electricity or the continuity equation. This latter refers to any one frame individually, and is derived from the equations of the field. Thus from (4) we obtain

$$\begin{aligned}\partial\rho/\partial t &= \partial(\Sigma\partial E_x/\partial x)/\partial t\\ &= \Sigma\partial(\partial E_x/\partial t)/\partial x\\ &= -\Sigma\partial(\rho v_x)/\partial x\end{aligned}$$

i.e., $$\partial\rho/\partial t + \Sigma v_x\partial\rho/\partial x + \rho\Sigma\partial v_x/\partial x = 0.$$

Those familiar with hydrodynamical equations will recognise that this equation accounts for any change in electric charge within a given surface by a flow of electricity across the surface, no creation or destruction being required. But this result has no direct bearing on the measures of charge made in different frames. It is true in any frame; but it could conceivably be true, even if S observers and S′ observers made different measures of charge. There are an infinite number of relations between charge measures in the two frames, other than that of equality, which would be consistent with conservation in each frame.

In passing, it is worth while noting two results, which can be easily verified by equations (7) and (8), viz., that

$$H^2 - E^2 = H'^2 - E'^2$$

and $$(\mathbf{E}\,.\,\mathbf{H}) = (\mathbf{E}'\,.\,\mathbf{H}'),$$

or the difference of the squares of the intensities of the magnetic and electric fields and their *geometric* product are invariant for any Lorentz transformation. The second one yields as a special result that if **E** and **H** are perpendicular to one another in S, then likewise **E′** and **H′** are perpendicular in S′.

We shall now turn to the question of the relativity of the "ponderomotive" forces on a charged body, as embodied in the equations (2) and (6) of the last chapter.

In the frame S the force on a body (idealised as a particle) with a charge e is $e(E_x + v_yH_z - v_zH_y)$ and two similar components. In the frame S′ it is $e(E_x' + v_y'H_z' - v_z'H_y')$, etc. We need not accent the symbol e, on account of the invariance of the charge just referred to. We have equations (7) and (8) as the relations between **E**, **H**, **E′**, **H′**, and the now familiar equations connecting **v** and **v′**.

Hence, if we write **F** for $e(\mathbf{E} + [\mathbf{v}\cdot\mathbf{H}])$,

$$\begin{aligned}\beta'F_x' &= e\beta'(E'_x + v_y'H_z' - v_z'H_y')\\ &= e\{\alpha(\beta - u\beta v_x)E_x + \beta v_y\alpha(H_z - uE_y) - \beta v_z\alpha(H_y + uE_z)\}\\ &= e\alpha\{\beta(E_x + v_yH_z - v_zH_y) - u\beta(v_xE_x + v_yE_y + v_zE_z)\}\end{aligned}$$

Now it is easy to show that

$$(\mathbf{v}\,.\,\mathbf{F}) = \Sigma v_xF_x = \Sigma v_xE_x.$$

Hence

$$\beta' F_x' = \alpha\{\beta F_x - u\beta(\mathbf{v} \,.\, \mathbf{F})\}.$$

Further,

$$\begin{aligned}\beta' F_y' &= e\beta'(E_y' + v_z' H_x' - v'_x H_z') \\ &= e\{\alpha(\beta - u\beta v_x)\alpha(E_y - uH_z) + \beta v_z H_x - \\ &\qquad\qquad \alpha(\beta v_x - u\beta)\alpha(H_z - uE_y)\} \\ &= e\alpha^2\beta(E_y - u^2 E_y + u^2 v_x H_z - v_x H_z) + e\beta v_z H_x \\ &= e\beta(E_y + v_z H_x - v_x H_z) \\ &= \beta F_y.\end{aligned}$$

Similarly,

$$\beta' F_z' = \beta F_z.$$

It now follows, even without any further use of (7) and (8), that

$$\beta'(\mathbf{v}' \,.\, \mathbf{F}') = \alpha\{\beta(\mathbf{v} \,.\, \mathbf{F}) - u\beta F_x\}.$$

Thus the relativity of the ponderomotive force *on a given charge* (or *per unit charge*) is proved.

It is interesting to note that although the ponderomotive force components per unit charge and its activity do not constitute a cogredient tetrad (that property being possessed by β times these quantities), the components and activity *per unit volume* of the charged body are cogredient. This is extremely easy to demonstrate. For the components and the activity per unit volume are $\rho(E_x + v_y H_z - v_z H_y)$, two similar components and $\rho(\mathbf{v} \,.\, \mathbf{E})$, i.e., $\rho F_x/e$, $\rho F_y/e$, $\rho F_z/e$, $\rho(\mathbf{v} \,.\, \mathbf{F})/e$. But e is invariant. Hence the cogrediency is proved if ρF_x, ρF_y, ρF_z, $\rho(\mathbf{v} \,.\, \mathbf{F})$, constitute a cogredient tetrad; and this is so since βF_x, βF_y, βF_z, $\beta(\mathbf{v} \,.\, \mathbf{F})$ are such a tetrad, and ρ/β is invariant.

In electromagnetic theory it is well known that there exist two functions, one a scalar function $\phi(x, y, z, t)$, the other a vector function $\mathbf{A}(x, y, z, t)$, such that

$$\left.\begin{array}{l} E_x = -\,\partial\phi/\partial x - \partial A_x/\partial t \\ \text{and two similar equations; and} \\ H_x = \partial A_z/\partial y - \partial A_y/\partial z \end{array}\right\} \quad . \quad . \quad (10)$$

and two similar equations or, in vector notation,

$$\left.\begin{array}{l} \mathbf{E} = -\operatorname{grad}\phi - \partial\mathbf{A}/\partial t \\ \mathbf{H} = \operatorname{curl}\mathbf{A} \end{array}\right\} \quad . \quad . \quad (10\text{A})$$

For since

$$\Sigma\partial H_x/\partial x = 0,$$

we see that H_x, H_y, H_z can be written as in the second group of equations (10).

Also, since

$$\partial E_z/\partial y - \partial E_y/\partial z = -\partial H_x/\partial t,$$

it follows that

$$\partial(E_z + \partial A_z/\partial t)/\partial y = \partial(E_y + \partial A_y/\partial t)/\partial z,$$

and two similar equations can be obtained, so that

$$E_x + \partial A_x/\partial t,\ E_y + \partial A_y/\partial t,\ E_z + \partial A_z/\partial t$$

are the partial differential coefficients with respect to x, y, z of some scalar function. If we call this function $-\phi(x, y, z, t)$, we have the first group of (10).

By putting these values for **E** and **H** in the equations (2) and (3), and making use of (4) and (5), we can show that A_x, A_y, A_z, and ϕ are solutions of the differential equations

$$\left.\begin{array}{l} \partial^2 A_x/\partial t^2 - \partial^2 A_x/\partial x^2 - \partial^2 A_x/\partial y^2 - \partial^2 A_x/\partial z^2 = \rho v_x \\ \text{and two similar equations in } A_y \text{ and } A_z\text{, and} \\ \partial^2\phi/\partial t^2 - \partial^2\phi/\partial x^2 - \partial^2\phi/\partial y^2 - \partial^2\phi/\partial z^2 = \rho \end{array}\right\} \quad (11)$$

or, in vector rotation,

$$\left.\begin{array}{l} \partial^2\mathbf{A}/\partial t^2 - \triangle\mathbf{A} = \rho\mathbf{v} \\ \partial^2\phi/\partial t^2 - \triangle\phi = \rho \end{array}\right\} \quad . \quad . \quad (11\text{A})$$

The function ϕ is usually called the scalar potential, and **A** the vector potential of the field, and the object in introducing them here is to demonstrate that they constitute another of those cogredient tetrads of which we have now accumulated a fair number. If A_x', A_y', A_z', ϕ' are the values of these potentials in the frame S′ at a given point-instant for which A_x, A_y, A_z, ϕ are the values in S, it is not difficult to show that

$$\left.\begin{array}{l} A_x' = \alpha(A_x - u\phi) \\ A_y' = A_y;\ A_z' = A_z \\ \phi' = \alpha(\phi - uA_x) \end{array}\right\} \quad . \quad . \quad . \quad (12)$$

In fact, equations (11) and the known cogrediency of ρv_x, ρv_y, ρv_z, ρ suggest equations (12); but reference to equations (7) and (8) justify the cogredient property of the potentials. For

$$\begin{aligned} E_x' &= -\partial\phi'/\partial x' - \partial A_x'/\partial t' \\ &= -\alpha(\partial\phi'/\partial x - u'\partial\phi'/\partial t) - \alpha(\partial A_x'/\partial t - u'\partial A_x'/\partial x) \\ &= -\partial\alpha(\phi' - u'A_x')/\partial x \quad - \partial\alpha(A_x' - u'\phi')/\partial t. \end{aligned}$$

Also $\qquad \partial\phi/\partial x - \partial A_x/\partial t = E_x = E_x'.$

This is consistent with putting

$$\phi = a(\phi' - u'A_x')$$
$$A_x = a(A_x' - u'\phi').$$

Further,

$$E_y' = -\partial\phi'/\partial y' - \partial A_y'/\partial t'$$
$$= -\partial\phi'/\partial y - a(\partial A_y'/\partial t - u'\partial A_y'/\partial x),$$

and

$$H_z' = \partial A_y'/\partial x' - \partial A_x'/\partial y'$$
$$= a(\partial A_y'/\partial x - u'\partial A_y/\partial t) - \partial A_x'/\partial y.$$

So
$$E_y = a(E_y' - u'H_z')$$
$$= - a\partial\phi'/\partial y - a^2\partial A_y'/\partial t + a^2u'\partial A_y'/\partial x$$
$$- a^2u'\partial A_y'/\partial x + a^2u'^2\partial A_y'/\partial t + au'\partial A_x'/\partial y$$
$$= - \partial a(\phi' - u'A_x') - \partial A_y'/\partial t./\partial y$$

But
$$E_y = - \partial\phi/\partial y - \partial A_y/\partial t,$$

and this is again consistent with

$$\phi = a(\phi' - u'A_x'),$$

and, in addition,

$$A_y' = A_y.$$

Proceeding in this way, we can show similarly that all the equations (7) and (8) are consistent with (12).

A case of some importance deserves special mention here. If a system of electrified bodies be at rest in a frame S′, there is a pure electrostatic field in that frame; ϕ' does not depend on time, and as there is no magnetic field A′ is zero. In fact, ϕ' is the ordinary electrostatic potential of the field. In the frame S the system is in motion with a velocity u parallel to OX. By using the equations reciprocal to (7) and (8), and remembering that $H_x' = H_y' = H_z' = 0$, we obtain for the field in S:

$$E_x = E_x',\ E_y = aE_y';\ E_z = aE_z'$$
$$H_x = 0,\ H_y = - auE_z',\ H_z = auE_y'.$$

It can be shown that the electric *intensity* in S is not derivable from a potential function by spatial differentiation. It is, of course, derivable from a vector and a scalar potential in the usual way, and we have by equations reciprocal to (12):

$$A_x = - au'\phi' = au\phi'$$
$$A_y = A_z = 0$$
$$\phi = a\phi'.$$

Hence

$$\begin{aligned} E_x &= -\,\partial\phi/\partial x - \partial A_x/\partial t \\ &= -\,\alpha\partial\phi'/\partial x - \alpha u\partial\phi'/\partial t \\ E_y &= -\,\alpha\partial\phi'/\partial y \\ E_z &= -\,\alpha\partial\phi'/\partial z, \end{aligned}$$

which proves the statement made above; for $\partial\phi'/\partial t$ is not zero. (ϕ' does not depend on t' of course, but it does on x', which is equal to $\alpha(x - ut)$.)

However, although **E** cannot be derived from a potential function, it is not difficult to see that the ponderomotive force *on a given charge* can be so derived. For its components per unit charge are

$$E_x,\ E_y - uH_z,\ E_z + uH_y,$$

which are respectively equal to

$$E_x',\ \alpha E_y'(1 - u^2),\ \alpha E_z'(1 - u^2)$$

or

$$E_x',\ E_y'/\alpha,\ E_z'/\alpha.$$

Now

$$E_x' = -\,\partial\phi'/\partial x',$$

and

$$\begin{aligned} \partial\phi'/\partial x &= \alpha(\partial\phi'/\partial x' - u\partial\phi'/\partial t') \\ &= \alpha\partial\phi'/\partial x'. \end{aligned}$$

Hence

$$\begin{aligned} E_x' &= -\,\alpha^{-1}\partial\phi'/\partial x \\ E_y'/\alpha &= -\,\alpha^{-1}\partial\phi'/\partial y' \\ &= -\,\alpha^{-1}\partial\phi'/\partial y \end{aligned}$$

and

$$E_z'/\alpha = -\,\alpha^{-1}\partial\phi'/\partial z.$$

Thus the ponderomotive force per unit charge is derivable from the potential function ϕ'/α, or ϕ/α^2 where ϕ is the scalar potential of the field in S. Thus we see that a force due entirely to a centre of attraction or repulsion, varying as the inverse square of the distance does not comply with the Relativity test for forces. This emerges from the demonstration that the ponderomotive force on an electrified body (calculated in terms of **E** and **H**) complies with the test, whereas a force calculated with **E** alone does not. It follows at once that we cannot bring Newton's law of gravitation unmodified under the principle of Relativity; for there is no obliging vector, analogous to the magnetic vector in electromagnetic theory, to assist the gravitational vector to achieve that desired end.

Appendix on Electromagnetic Phenomena in a Moving Material Medium.

The equations of the field which were written down in the earlier part of this chapter (not containing the symbol ρ) were

originally developed by Maxwell for propagation *in vacuo*. The equations (2), (3), (4), (5) were used by Lorentz, and applied by him to his attempts to explain electromagnetic phenomena in material media in terms of the hypothesis of electrons (i.e., electrons in the wide sense of minute electrified particles of which atoms of matter are constituted in whole or in part). In these equations the field at any definite point would be subject to extremely rapid and fortuitous oscillations, consequent on the fortuitous vibrations of molecules, atoms, and electrons as pictured in the kinetic theory of matter. This would be true even in electrostatic and magnetostatic phenomena as ordinarily understood. But in discussions of electric and magnetic phenomena on the basis of experimental results, in which no consideration is paid to theories of the structure of matter, we generally regard matter as continuous, its density being obviously an average in which the irregularities due to granular structure are " smoothed out." Pressure and temperature are also familiar examples of this smoothing out process applied to irregularly varying molecular momenta and energy. In the same way electric intensity and magnetic intensity, as derived from actual measurements on a finite portion of matter, must be each an average of the values existing at every point within a finite, though small, volume for a finite, though small, time. In the following discussion **E**, **H**, and ρ represent such smoothed-out values for a physically small volume surrounding a point, and not actual values at a point, as was implied in the Lorentz equations of the electron theory given above. In fact, the equations we are going to write down are usually referred to as " macroscopic," in contrast to the term " microscopic," applied to the former. Maxwell was the first to suggest such equations, being guided to them by his equations for vacuum. He introduced, as well as **E** and **H**, two symbols representing electric induction, or " displacement," and magnetic induction, **D** and **B**. These will be familiar to every student of electricity and magnetism. These two vectors are related to the former two by the equations

$$\mathbf{D} = \epsilon\mathbf{E} \quad . \quad . \quad . \quad . \quad (13)$$

$$\mathbf{B} = \mu\mathbf{H} \quad . \quad . \quad . \quad . \quad (14)$$

or in Cartesians,

$$D_x = \epsilon E_x, \text{ etc.} \quad . \quad . \quad . \quad (13\text{A})$$

$$B_x = \mu H_x, \text{ etc.} \quad . \quad . \quad . \quad (14\text{A})$$

where ϵ and μ are the values of the specific electric and magnetic inductivities of the material medium. (The units are

again Heaviside-Lorentz units, introduced with the object of removing the inconvenient factor 4π from the most important equations.) It is understood that the medium is isotropic and at rest. To give as great generality as possible, it is assumed that the medium has also a finite electric conductivity σ, so that if all convective and conductive currents across a small area within the medium are smoothed out, we derive a current-density per unit area normal to the flow denoted by $\mathbf{j}$ (j_x, j_y, j_z) which is related to $\mathbf{E}$ by

$$\mathbf{j} = \sigma\mathbf{E} \quad . \quad . \quad . \quad . \quad (15)$$

or

$$j_x = \sigma E_x, \text{ etc.} \quad . \quad . \quad . \quad (15\text{A})$$

The Maxwell equations are then :

$$\partial D_x/\partial t + j_x = \partial H_z/\partial y - \partial H_y/\partial z \quad . \quad . \quad (16)$$

and two similar equations,

$$-\partial B_x/\partial t = \partial E_z/\partial y - \partial E_y/\partial z \quad . \quad . \quad . \quad (17)$$

and two similar equations,

$$\Sigma\partial D_x/\partial x = \rho \quad . \quad . \quad . \quad . \quad (18)$$

$$\Sigma\partial B_x/\partial x = 0 \quad . \quad . \quad . \quad . \quad (19)$$

or

$$\partial\mathbf{D}/\partial t + \mathbf{j} = \text{curl } \mathbf{H} \quad . \quad . \quad . \quad (16\text{A})$$

$$-\partial\mathbf{B}/\partial t = \text{curl } \mathbf{E} \quad . \quad . \quad . \quad . \quad (17\text{A})$$

$$\text{div } \mathbf{D} = \rho \quad . \quad . \quad . \quad . \quad (18\text{A})$$

$$\text{div } \mathbf{B} = 0 \quad . \quad . \quad . \quad . \quad (19\text{A})$$

Equations (16) to (19) are called the field equations. Equations (13), (14), (15) are called the constitutive equations.

These are the equations for a medium "at rest," and as usual the question arises: to what frame? In pre-relativity days the ether was the frame, and Maxwell, Hertz, Larmor, and Lorentz all suggested generalisations of the equations which would cover the case of matter in motion through the ether. The work of Lorentz was extremely exhaustive, and being based on his microscopic equations for the electron theory, it is not surprising to find that his macroscopic equations can be brought within the bounds of the Relativity principle, since it has just been shown that the microscopic equations pass the Relativity test. But, of course, to Lorentz the problem was to transform from the equations (13) to (19), supposed true for matter at rest in the ether, to a form suitable for matter in motion through the ether. To the relativist, the problem is to discover the field equations and constitutive equations which, assumed

to be true in frame S, will transform to exactly the same form in S′ on the basis of the Lorentz transformation of the space-time co-ordinates, and to find the relations between the accented and unaccented symbols which will comply with this demand. It was Minkowski who first gave a complete solution to this problem ; complete in the sense that he gave a set of equations which satisfied the Relativity test, but left open to some extent the precise physical quantity which a given symbol was supposed to represent. But if one employs the method used in the chapter just concluded, it can be shown that equations (16) to (19) transform into an exactly similar group in accented symbols, provided

$$\left.\begin{aligned} E_x' &= E_x \\ E_y' &= a(E_y - uB_z) \\ E_z' &= a(E_z + uB_y) \end{aligned}\right\} \quad . \quad . \quad . \quad (20)$$

$$\left.\begin{aligned} D_x' &= D_x \\ D_y' &= aD(_y - uH_z) \\ D_z' &= a(D_z + uH_y) \end{aligned}\right\} \quad . \quad . \quad . \quad (21)$$

$$\left.\begin{aligned} H_x' &= H_x \\ H_y' &= a(H_y + uD_z) \\ H_z' &= a(H_z - uD_y) \end{aligned}\right\} \quad . \quad . \quad . \quad (22)$$

$$\left.\begin{aligned} B_x' &= B_x \\ B_y' &= a(B_y + uE_z) \\ B_z' &= a(B_z - uE_y) \end{aligned}\right\} \quad . \quad . \quad . \quad (23)$$

$$\left.\begin{aligned} j_x' &= a(j_x - u\rho) \\ j_y' &= j_y \,;\; j_z' = j_z \\ \rho' &= a(\rho - uj_x) \end{aligned}\right\} \quad . \quad . \quad . \quad (24)$$

Suppose now that a material medium is at rest in S′, and therefore in motion through S with a velocity u parallel to OX. As it is at rest in S′, the simple constitutive equations (13), (14), (15) are supposed to be true, i.e.,

$$\begin{aligned} \mathbf{D}' &= \epsilon\mathbf{E}' \\ \mathbf{B}' &= \mu\mathbf{H}' \\ \mathbf{j}' &= \sigma\mathbf{E}' \end{aligned}$$

for these are known to be true in the case of stationary, homogeneous, and isotropic media, ϵ, μ, and σ being the values which would be found for the constants by ordinary laboratory experiments (in Lorentz units, of course).

It follows that for the body in motion we have

$$D_x=\epsilon E_x\,;\ D_y-uH_z=\epsilon(E_y-uB_z)\,;\ D_z+uH_y=\epsilon(E_z+uB_y) \quad (25)$$

$$B_x=\mu H_x\,;\ B_y+uE_z=\mu(H_y+uD_z)\,;\ B_z-uE_y=\mu(H_z-uD_y) \quad (26)$$
$$\alpha(j_x-u\rho)=\sigma E_x\,;\ j_y=\sigma\alpha(E_y-uB_z)\,;\ j_z=\sigma\alpha(E_z+uB_y) \quad (27)$$

Let us now restrict the body to be a dielectric, so that $\sigma = 0$ and therefore $\mathbf{j} = 0$, and let it be in the form of a slab between two metal plates, which are connected to the terminals of an electrometer and are parallel to the plane OXY. Suppose a permanent magnetic field is established parallel to the axis of y, so that $H_x = H_z = B_x = B_z = 0$, and in consequence the slab is moving with a velocity u at right angles to $\mathbf{H}$. The equations show, and experiment verifies, that there must be an electric field and polarisation of the dielectric produced in the direction OZ, i.e., perpendicular to the directions of the motion and the magnetic field. For, by (25),

$$D_z + uH_y = \epsilon(E_z + uB_y)$$
and
$$D_x = D_y = E_x = E_y = 0$$

By (26) $$B_y + uE_z = \mu(H_y + uD_z).$$

Hence $$D_z + uH_y = \epsilon E_z + \epsilon\mu(uH_y + u^2D_z) - \epsilon u^2 E_z$$
or $$(1 - \epsilon\mu u^2)D_z = \epsilon(1 - u^2)E_z + u(\epsilon\mu - 1)H_y.$$

In the experimental work the apparatus is not sufficiently refined to take account of terms involving the square of u; so neglecting these, we have

$$D_z = \epsilon E_z + u(\epsilon\mu - 1)H_y \quad . \quad . \quad . \quad (28)$$

If V is the difference of potential established between the plates and read on the electrometer,

$$E_z = V/d$$

where d is the distance between the plates.

Also D_z is the surface density of the charge on the plates, and is therefore equal to KV where K is the capacity of the plate condenser per unit area. Hence

$$(K - \epsilon/d)V = u(\epsilon\mu - 1)H_y.$$

If the permeability of the medium is unity, we have

$$(K - \epsilon/d)V = u(\epsilon - 1)H_y.$$

This equation was verified in a well-known experiment by H. A. Wilson in 1904 (" Phil. Trans.," A, 204, p. 121). This was before the appearance of Einstein's first paper on Relativity,

and the experiment was really designed to decide between the above equation, which had been derived by Lorentz from his theory, and an equation which had been obtained previously by Hertz, using a different field equation for moving matter. Hertz's result was

$$(K - \epsilon/d)V = u\epsilon H_y.$$

Wilson showed conclusively that the factor on the right-hand side was $\epsilon - 1$ and not ϵ, thus disproving Hertz's equations.

As a matter of fact, Lorentz's equation is not quite (28) above, but is

$$D_z = \epsilon E_z + u(\epsilon - 1)\mu H_y.$$

As μ is practically unity for any dielectric, experiment fails to discriminate between this and the genuine Relativity result.

CHAPTER V.

We have now become familiar with the most arresting feature of the analysis contained in the preceding pages, viz., the dependence of the time co-ordinate in one frame of reference on both the time and space co-ordinates in another frame. Another aspect of this is the intertwining of the measures of momentum and energy, of force and activity. An equally striking property of the theory is the derivation of the electric or magnetic vector in one frame from both the electric and magnetic vectors in another. Every physicist is familiar from the days of his first lessons in Mechanics with the fact that for the sake of mathematical analysis we resolve measurements of displacement, velocity, acceleration, force into components along directions which are entirely arbitrary. What is in reality an indivisible unity in our space, we divide up in a purely artificial manner for our own convenience, our justification being that on subjecting each portion to suitable mathematical treatment and reuniting the results, we arrive at a reality of our perceptual experience. Nevertheless, although the power of analysis by Cartesian methods is admitted beyond all question, the mathematical physicist has of late years been resorting more and more to the use of the methods of Vectorial Analysis, inasmuch as they keep well in view the unity which is apt to disappear when resolution into components takes place at the outset of the solution of some problem.

Now one illustration of this underlying unity can be seen in the fact that when for some purpose or other we transform our axes, i.e., resort to a different mode of resolving, any one component in the new axes depends in general on all the components in the old system, and not merely on one. But this is just the feature we have signalised above in the analysis of Relativity; there it occurs not alone in connection with the space co-ordinates; it exhibits itself in connection with both space and time co-ordinates.* Is it not possible then that in accepting

* The fact that only one space co-ordinate appears in the most novel of the equations of the Lorentz transformation as used so far, depends on the special choice we have made for the axis of x. It is a feature which is of little importance and will be removed shortly.

the truth of the Relativity standpoint, we may find ourselves driven in the long run to admit also that Space and Time are not distinct and separate unities in themselves? May they not be two divisions of some underlying unity, which each of us separates in manner most convenient to ourselves, no one method of separation, however, being of greater fundamental importance than any other?

The concept of multi-dimensional space is not a novelty, and although we cannot visualise it, we frequently make use of its geometric terminology as a convenience and a means of economising speech and writing when dealing with the measurable properties of physical systems—notably so in the subjects of Generalised Dynamics and Statistical Mechanics. In particular, the existence of a fourth dimension has always appealed to the imagination of the scientific worker and philosopher. The notion that our arrangement of events in time order is a three-dimensional being's mode of dealing with a static arrangement in space of four dimensions which would actually be perceived as such by a four-dimensional being, is as old as Greek philosophy. What we perceive at any instant is a part of a three-dimensional section of a four-dimensional world, the section being "normal" to the "axis of time." Even when at rest in our space, any object is moving parallel to the axis of time. When an object moves in our three-dimensional space, it has a component of motion along the axis of time as well as along any spatial axes which we select. We may not go so far as to postulate the actual existence of such a four-dimensional space-time continuum, yet we make practical use of the idea every time we draw a graph representing displacement of a body and time. There, of course, we are dealing with a one-dimensional movement and time, and the paper or blackboard surface is two-dimensional. If practicable, we could represent the graph of a two-dimensional motion and time in a three-dimensional frame, say by threads instead of chalk or pencil-marks; but these are, after all, only material aids to our geometry. We can easily visualise the curves or graphs, because we can perceive three dimensions. When we attempt the same procedure for three-dimensional motion and time, our geometric intuitions fail because we do not perceive four dimensions. We have to resort to analogy. We assume that there is an axis orthogonal to any three mutually perpendicular axes which we choose in our space; that the origin of our space axes is travelling along this fourth axis of time, but that there is a "fixed" origin for our four-

dimensional axes, viz., the position occupied in this four-dimensional world by the origin of our space axes at some definite instant in the history of the events we are considering. It has become customary since Minkowski introduced the term to refer to this four-dimensional concept as "the World" ("*die Welt*"), retaining the word, space, for our ordinary three-dimensional perceptions. "Space-time continuum" is also used. In the World the history of a particle is contained in the curve which is the locus of all its positions in the World. If at rest in our space, this is a straight line in space-time parallel to the time axis; if in uniform motion in our space, it is a straight line making a constant angle with the time-axis; if in uniformly accelerated motion in our space, it is a parabola with the tangent at its vertex parallel to the time-axis, etc.

Similarly, the history of a body is contained in a four-dimensional tubular space generated by the "World lines," "Space-time lines," of its various particles. When one experiences any difficulty in imagining this World, an appeal to the analogy of the three-dimensional World of a two-dimensional space will prove very helpful. One fact about this four-dimensional World we must grasp and bear in mind for our immediate purpose: just as in three-dimensional geometry the normals to a plane are all parallel to each other (if the geometry be Euclidean) and intersect the plane each in one point only, so in four-dimensional geometry, a given three-dimensional section has its normals parallel to each other, i.e., there is one and only one direction in space-time perpendicular to all the lines in a given space, and any such normal cuts this space in only one point, all other points of the normal not lying in this space.

As stated, these ideas are not novelties; but their special importance for Relativity lies in the facts mentioned at the beginning of this chapter. In pre-relativity days one regarded the space sections of the World made by different observers as "parallel" to one another, i.e., as normal to one direction in the World which would be the direction of the common axis of time. This appears in considering the Newtonian transformation for uniform relative velocity between the space-frames of two observers:

$$x_1' = x_1 - ux_4$$
$$x_2' = x_2\,;\ \ x_3' = x_3$$
$$x_4' = x_4$$

(where the suffixes 1, 2, 3 refer to space co-ordinates and 4 to

the time co-ordinates). We could make a still further transformation to space axes obtained by rotating the three space-axes OX_1', OX_2', OX_3' about some line in space, but we should still retain

$$x_4' = x_4.$$

We can state this Newtonian procedure in a picturesque way by saying that all observers make the same space section (at a given moment) of the World, no matter what their relative velocity may be, or, in another way, by saying that the World lines of any two bodies at relative rest in space are parallel to one another and are in the direction of the time axis. But for the relativist each observer has his own axis of time in the World. Another observer, moving *in space* relative to the former, employs a different time-axis inclined to that of the former and so makes a different series of space sections of the World when thinking intelligently about his experiences. Of course, these geometrical statements are contained in the analytical statement of the Lorentz transformation:

$$\begin{aligned} x_1' &= a(x_1 - ux_4) \\ x_2' &= x_2\,;\ x_3' = x_3 \\ x_4' &= a(x_4 - ux_1). \end{aligned}$$

On consideration, it will appear that the space co-ordinates and the time co-ordinate in a frame S are related to those in S′, just as the four co-ordinates of a point in the World referred to axes OX_1, OX_2, OX_3, OX_4, are related to the four co-ordinates of the same point referred to axes OX_1', OX_2, OX_3, OX_4', two of the axes remaining unchanged, the other two being rotated in their common plane, and the units in which the first and fourth co-ordinates are measured being changed. In the figure the axes OX_1, OX_4 cut the conjugate rectangular hyperbolæ

$$x_1^2 - x_4^2 = \pm a^2$$

in the points A_1, A_4. The lines OX_1' and OX_4' are a pair of conjugate diameters inclined to the lines OX_1, OX_4 at an angle θ given by

$$\tan\theta = u.$$

These cut the branches in A_1', A_4'.

The equations of the hyperbolæ referred to OX_1', OX_4' are, as is well known,

$$x_1'^2 - x_4'^2 = \pm a'^2,$$

where $$a' = OA_1' = OA_4',$$

and it is not difficult to show that

$$OM'/a' = \alpha(OM/a - u \,.\, MP/a)$$
$$M'P/a' = \alpha(MP/a - u \,.\, OM/a),$$

where $\alpha = \cos\theta/\sqrt{\cos 2\theta} = 1/\sqrt{(1 - \tan^2\theta)} = 1/\sqrt{(1 - u^2)}$.

Hence if we write

$$x_4 = OM/a\,;\ x_1 = MP/a$$
$$x_4' = OM'/a'\,;\ x_1' = M'P/a',$$

i.e , employ a as unit of length and time in one frame, and a' as unit in the other, we reproduce the Lorentz transformation.

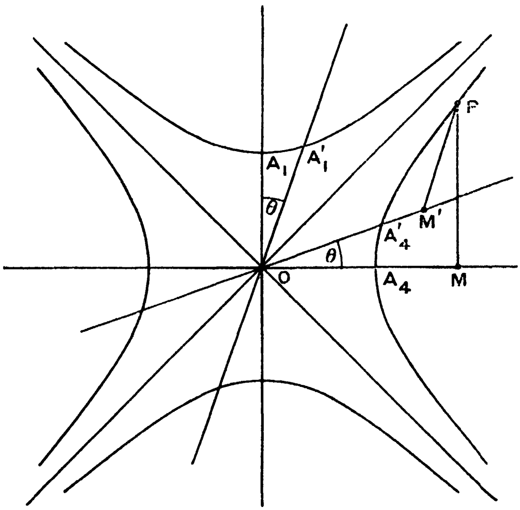

FIG. 4.

The two axes OX_1' and OX_4' would still be perpendicular to OX_2 and OX_3, but not perpendicular to each other, that is, from the point of view of the observers in S. Of course, observers in S′ would consider all their axes as mutually orthogonal, but the axes of x_1 and x_4 employed by S as not perpendicular to each other. This is a fanciful way of stating that units of length and time are different, and, further, that simultaneity is a relative conception. To give ourselves a chance of visualising this, consider a two-dimensional space with particles moving in it. The World will now be a three-dimen sional one, and the World lines of these particles will be defir·

curves in it. A plane parallel to OX_1X_2 will cut these World lines in points which represent simultaneous positions of the particles from the point of view of one observer. From the point of view of an observer moving relative to the first with a velocity u parallel to OX_1, simultaneous positions of the particles will be represented by the intersections of the World lines with planes parallel to $OX_1'X_2$.

There is, of course, no necessity to restrict ourselves to an alteration in only two of the axes. In the earlier chapters dealing with uniform relative motion we did so, and gave a fictitious importance to the axis of x, which it assumed because we chose it to be parallel to the direction of relative motion of the frames S and S′. But, after all, we are quite familiar with changes of axes in space, and for any given time axis in the World we can have an infinite number of sets of space axes, viz., any triad of lines in our space, or, rather, the positions of these lines in the World at a definite original instant. It is easy to remove the importance of OX. Thus, reverting to ordinary notation for a little, we choose axes in S such that the direction cosines with respect to them of the line OO′ (i.e., the line of relative motion of S′ to S), and two lines at right angles to it and to each other are (l_1, m_1, n_1), (l_2, m_2, n_2), (l_3, m_3, n_3). The axes in S′ are such that these same three lines have direction cosines (l_1', m_1', n_1'), (l_2', m_2', n_2'), (l_3', m_3', n_3'). Then the first three of the Lorentz equations become

$$\begin{aligned} l_1'x' + m_1'y' + n_1'z' &= \alpha(l_1x + m_1y + n_1z - ut) \\ l_2'x' + m_2'y' + n_2'z' &= l_2x + m_2y + n_2z \\ l_3'x' + m_3'y' + n_3'z &= l_3x + m_3y + n_3z. \end{aligned}$$

Solving for x', y', z', we obtain

$$\begin{aligned} x' = (\alpha l_1'l_1 + l_2'l_2 + l_3'l_3)x &+ (\alpha l_1'm_1 + l_2'm_2 + l_3'm_3)y \\ &+ (\alpha l_1'n_1 + l_2'n_2 + l_3'n_3)z - \alpha u l_1't, \end{aligned}$$

and two similar equations.

If we chose axes so that

$$l_r' = l_r,\ m_r' = m_r,\ n_r' = n_r,\ (r = 1, 2, 3),$$

these equations simplify to:

$$\left.\begin{aligned} x' &= (\overline{\alpha-1}\cdot l_1^2+1)x+(\alpha-1)l_1m_1y+(\alpha-1)l_1n_1z-\alpha u_xt \\ y' &= (\alpha-1)m_1l_1x+(\overline{\alpha-1}\cdot m_1^2+1)y+(\alpha-1)m_1n_1z-\alpha u_yt \\ z' &= (\alpha-1)n_1l_1x+(\alpha-1)\cdot n_1m_1y+(\overline{\alpha-1}\cdot n_1^2+1)z-\alpha u_zt \end{aligned}\right\} \quad \text{(A)}$$

ere u_x, u_y, u_z are components of u along the axes.

In connection with this latter choice of axes, it must be noted that they do not coincide at $t = 0 = t'$, e.g., if $t = 0$, the point (1, 0, 0) in S, which lies on OX, does not lie on O′X′, for its co-ordinates in S′ are $(a - 1)l_1^2 + 1$, $(a - 1)m_1l_1$, $(a - 1)n_1l_1$. This, in fact, illustrates an earlier remark that a pair of lines which are at right angles from the point of view of S are not in general at right angles from the point of view of S′, and so could not be simultaneously (to S′ observers) coincident with a pair of lines which are perpendicular to each other from the point of view of S′.

The fourth of the Lorentz equations is easily obtained for general axes. It is

$$t' = a\{t - u(l_1x + m_1y + n_1z)\}$$

or

$$t' = -au_xx - au_yy - au_zz + at \qquad \text{(B)}$$

In vectorial notation equations (A) and (B) can be written as follows, where $\mathbf{r}$ and $\mathbf{r}'$ represent the radii vectors OP and O′P′:

$$\mathbf{r}' = \mathbf{r} - at\mathbf{u} + (a - 1)(\mathbf{u}\,.\,\mathbf{r})\mathbf{u}/u^2 \qquad \text{(C)}$$

$$t' = a\{t - (\mathbf{u}\,.\,\mathbf{r})\} \qquad \text{(D)}$$

In (C), $(\mathbf{u}\,.\,\mathbf{r})\mathbf{u}/u^2$ represents a vector in the direction of $\mathbf{u}$ having the same magnitude as the resolved part of $\mathbf{r}$ in the direction of $\mathbf{u}$.

The reciprocal equations are:

$$\mathbf{r} = \mathbf{r}' + at'\mathbf{u} + (a - 1)(\mathbf{u}\,.\,\mathbf{r}')\mathbf{u}/u^2 \qquad \text{(E)}$$

$$t = a\{t' + (\mathbf{u}\,.\,\mathbf{r}')\} \qquad \text{(F)}$$

To return once more to a consideration of the space-time World, we see that the more general Lorentz equations express a general rotation of all the axes, and are not like the simpler form of the equations, concerned with a rotation "in the plane" of the time axis and one space axis. The essential feature of any of the forms, simple or general, as contrasted with any forms of the Newtonian transformation equations, is the implication that the axis of time is in different directions in the World for observers in relative motion to each other in space, i.e., that their space sections are different, which is our geometrical way of saying that a group of simultaneous events for one observer cannot in general contain all the events in a group of events simultaneous to the other observer.

There is one feature about the rotation which cannot escape notice. As far as the space axes are concerned, they experience a rotation as a rigid frame and remain at right angles to one

another when we use the simple Lorentz transformation, while the time-axis suffers a rotation with respect to one of the space-axes, which, however, maintains it in a certain geometric relation to that space axis, viz., as a diameter of a certain rectangular hyperbola conjugate to the space-axis. In the more general transformation, none of the axes preserve the relation of orthogonality, but they preserve the relation of a conjugate tetrad of diameters to either of the four-dimensional hyperquadrics

$$x_1^2 + x_2^2 + x_3^2 - x_4^2 = \pm 1.$$

For those who are unfamiliar with the concept of four dimensions, the appeal to analogy is again helpful, and it will be found that if we limit our equations to a two-dimensional space and a three-dimensional World, the equation

$$x_1^2 + x_2^2 - x_4^2 = \pm 1$$

transforms into

$$x_1'^2 + x_2'^2 - x_4'^2 = \pm 1,$$

and if OX_1, OX_2, OX_4 are a set of conjugate axes for these hyperboloids, then so also are OX_1', OX_2', OX_4'. In our four-dimensional World it is the mutual conjugacy of the four axes with regard to a definite hyperboloidal hyper-surface which is the invariant property of any one of the sets of axes connected by a general Lorentz transformation. When we say that in stepping from one frame to another in space, we rotate our four-dimensional axes, we imply that the rotation is of this nature, and not a rotation analogous to the rigid rotation which preserves orthogonality in three dimensions. The objection may be urged that the three space-axes have been chosen to be mutually orthogonal; that is so—to observers at rest relative to them. Each group of observers chooses orthogonal space axes to suit their own convenience, but such axes are not in general orthogonal to other observers in relative motion; that feature was emphasised above. Appealing to geometry of three dimensions once more for an analogue, we note that any plane does not necessarily intersect two planes perpendicular to each other in perpendicular lines; a certain limited class of planes does so, but not all planes.

The analysis of the general Lorentz transformation is the analysis of a hyperboloidal hyper-surface in a four-dimensional space referred to any four conjugate diameters as axes. This arises from the fact that with any Lorentz transformation we have

$$x_1^2 + x_2^2 + x_3^2 - x_4^2 = x_1'^2 + x_2'^2 + x_3'^2 - x_4'^2;$$

or

$$x_1^2 + x_2^2 + x_3^2 - x_4^2$$

is invariant; and the minus sign before the last term introduces the analysis suitable for hyperbolic figures. Had the plus sign occurred before all the terms, we should have been dealing with the analysis of a spherical hyper-surface.

It is, however, possible to get rid of the minus sign and give the analysis a formal similarity to the analysis of orthogonal transformation in four dimensions. If, instead of writing x_4 for the time co-ordinate, we put x_4 equal to the product of $\sqrt{(-1)}$ and the time co-ordinate ("imaginary time"), we see that

$$x_1^2 + x_2^2 + x_3^2 + x_4^2$$

remains invariant for a general Lorentz transformation. Thus the simple transformation becomes

$$\begin{aligned} x_1' &= ax_1 + \iota aux_4 \\ x_2' &= x_2; \quad x_3' = x_3 \qquad \{\iota = \sqrt{(-1)}\} \\ x_4' &= ax_4 - \iota aux_1, \end{aligned}$$

so that if we introduce an imaginary angle ϕ, for which

$$\begin{aligned} \tan\phi &= \iota u \\ \sec\phi &= \sqrt{(1 - u^2)} \\ \cos\phi &= a \\ \sin\phi &= \iota au, \end{aligned}$$

we can write

$$\begin{aligned} x' &= x_1 \cos\phi + x_4 \sin\phi \\ x_2' &= x_2; \quad x_3' = x_3 \\ x_4' &= -x_1 \sin\phi + x_4 \cos\phi, \end{aligned}$$

which corresponds to a rotation of the axes OX_1, OX_4 in their own plane through an angle ϕ, the axes remaining at right angles.

The general Lorentz transformation corresponds to a rotation of all the axes in the World in which the space-axes and axis of imaginary time remain orthogonal. The most convenient way of writing it is:

$$\left.\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 \\ x_2' &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 \\ x_3' &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 \\ x_4' &= a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4 \end{aligned}\right\} \quad . \quad (1)$$

where the a coefficients are constants depending on the relative movement of the space frames considered, and can be easily

interpreted from the earlier expressions, remembering that x_4 now is equal to ιx_4 in the former equations. It will be seen that the nine coefficients involving the digits 1, 2, 3 in their suffixes are real, the six involving the digit 4 once in their suffixes are imaginary, while a_{44} is real and equal to a, or $(1 - u^2)^{-\frac{1}{2}}$.

But for further progress there is no need to make use of the values of the a coefficients in terms of u and the direction cosines.

Since

$$\Sigma x_a{}^2 = \Sigma x_a'^2,$$

it follows that

$$a_{11}{}^2 + a_{21}{}^2 + a_{31}{}^2 + a_{41}{}^2 = 1 \quad . \quad . \quad (2)$$

and three similar equations, and

$$a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32} + a_{41}a_{42} = 0 \quad . \quad . \quad (3)$$

and five similar equations—ten equations in all.

It considerably economises labour to introduce the summation symbol and write (1) as

$$x_\lambda' = \sum_a a_{\lambda a} x_a,$$

where the summation extends over the values 1, 2, 3, 4 for a; putting λ equal to 1, 2, 3, 4 in succession, we have the four equations (1). Indeed, since the suffix a appears twice in the term which has to be summed, we can omit the symbol Σ on the understanding that any suffix which appears twice in a term on either side of an equation is a "dummy," i.e., the term has to be summed for values 1, 2, 3, 4 of the repeated suffix. We shall for convenience employ the earlier letters of the Greek alphabet, such as α, β, ϵ for dummy suffixes, and later letters such as κ, λ, μ for suffixes which are not dummy.

Thus we write equations (1) in the concise form

$$x_\lambda' = a_{\lambda a} x_a \quad . \quad . \quad . \quad . \quad (1)$$

and conditions (2) and (3) in the form

$$\begin{aligned} a_{a\lambda}a_{a\mu} &= 1 \text{ if } \lambda = \mu \quad . \quad . \quad . \quad (2) \\ &= 0 \text{ if } \lambda \neq \mu. \quad . \quad . \quad . \quad (3) \end{aligned}$$

Arising out of equations (2) and (3), the following results are true and not difficult to establish:

Theorem I.—The determinant,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$$

is equal to unity.

Theorem II.—Any constituent of the determinant is equal to its co-factor.*

Theorem III.—Any second minor of the determinant is equal to its co-factor.

Theorem IV.—Equations similar to (2) and (3) are true if the determinant is read by rows instead of columns, that is

$$a_{\lambda\alpha}a_{\mu\alpha} = 1, \text{ if } \lambda = \mu \quad . \quad . \quad . \quad (4)$$

$$= 0, \text{ if } \lambda \neq \mu \quad . \quad . \quad . \quad (5)$$

and, further,

$$x_\lambda = a_{\alpha\lambda}x_\alpha'. \quad . \quad . \quad . \quad . \quad (6)$$

(*N.B.*—$a_{\alpha\lambda}$ is not to be confused with $a_{\lambda\alpha}$; it is not in general true that $a_{\lambda\mu} = a_{\mu\lambda}$.)

From this point onwards let us accept the lead which has been given to us by our equations, and admit the reality of the space-time continuum. The objection that it cannot be "perceived" is trivial. Neither can space or time be "perceived" in the sense which is implied in that use of the word "perception." Space and time are modes of perceiving things. Relativity has taught us that the two modes are not independent

* The sign of the co-factor must be carefully noted. Thus the co-factor of a_{11} is the first minor obtained by removing the first row and first column, but the co-factor of a_{23} is *minus* the first minor obtained by removing the second row and third column, because in order to bring the constituent a_{23} into the leading place we must first of all displace the second row into the first row by one step and then the third column into the first column by two steps, three steps in all, i.e. an *odd* number. If the steps required in any case are even in number the co-factor is the first minor with sign unchanged.

Similar adaptation of the sign must be made in the case of the co-factors of second minors. Thus taking the second minor

$$\begin{vmatrix} a_{22} & a_{24} \\ a_{32} & a_{34} \end{vmatrix}$$

its co-factor is

$$-\begin{vmatrix} a_{11} & a_{13} \\ a_{41} & a_{43} \end{vmatrix}$$

for it requires one step to transfer the second column to the first and then two steps to transfer the fourth column to the second; another step brings the second row to the first, and still another the third row to the second. Thus five steps in all (an odd number) brings the first of the second minors written above into the leading place.

of one another. The manner in which they depend on one another is special to each observer; but for human beings, on a limited portion of the earth's surface, who use nearly the same frame of reference in life's ordinary affairs, the time axes are so nearly parallel that all their space sections are practically the same. Hence arises the illusion of the independence of space and time. The question is not one of philosophical interest alone. We have just seen the elegant and symmetrical form into which the Lorentz equations can be thrown, a form in which any distinctive feature of the time co-ordinate as such disappears. But this is no accident. Not only these initial equations, but all the equations of the Restricted Relativity theory can be exhibited in a similar elegant form which is a perfect analogue in *four* variables of the Analytical Geometry and Statics of the text-books. For this purpose we employ the branch of mathematical analysis known as "Tensor Analysis." If its sole use were to provide a pretty garment for the restricted theory of Relativity, the critic might well say that the mathematician had allowed to his æsthetic sense an undue importance. But as we proceed further we shall see that the great stride from the Special to the General Relativity theory, which has created such a revolution in our ideas about gravitation and also electromagnetism, could hardly have been made, certainly not with such powerful effect, had this branch of analysis not been ready to hand. Indeed, the very generalisation of it which is required in its application to the general relativity of all phenomena, has not only guided us to the formation of *geometrical* concepts concerning the four-dimensional world, but is actually impelling us towards researches, which may have as their outcome success in the measurement of such geometrical properties. With such ideas and possibilities before man's mind, it is idle for the critic to refer to the reality of space-time as a pointless speculation.

We shall, therefore, in the next two chapters, give an exposition of Tensor Analysis as required for the special theory of Relativity. This will lead us naturally to the second part, where we shall proceed to develop it in connection with the idea of General Relativity, which abandons the restriction that all frames of reference must be in *uniform* relative motion to each other. In all this, however, we shall keep physical phenomena, as such, well in view. When familiarity with the mathematical weapon has been assured, the final chapters will put before the reader the thought that all phenomena are but some aspect of a "World Geometry."

CHAPTER VI.

VECTORS.

ANY of the usual directed quantities or vectors in Physics is said to be represented by a point in space, if the displacement from an origin O to the point has a direction parallel to the direction of the quantity and a length proportional to its magnitude; so that the components of a vector are representable by the co-ordinates of a point. In consequence, a change from one set of rectangular axes to another set with the same origin involves relations between the new components of a vector and the old which are entirely analogous to equations (1) of the last chapter (the suffixes being confined to 1, 2, 3), provided the coefficients are connected by equations analogous to (2), (3), (4), (5).

In the same way, if we have any group of four quantities which on transformation are related as (x_1', x_2', x_3', x_4') are related to (x_1, x_2, x_3, x_4) by equations (1), we refer them as components of a vector in four dimensions, a "four-vector." If P_1, P_2, P_3, P_4 are the components of a four-vector **P** along the axes OX_1, OX_2, OX_3, OX_4, and P_1', P_2', P_3', P_4' are the components of the same vector along OX_1', OX_2', OX_3', OX_4', then

$$P_1'^2 + P_2'^2 + P_3'^2 + P_4'^2 = P_1^2 + P_2^2 + P_3^2 + P_4^2,$$

or $P_\alpha P_\alpha$ is invariant.

This is a particular case of a general theorem (which can be easily proved by an appeal to equations (2)-(5)) that if **P** and **Q** are two four-vectors, then

$$P_\alpha Q_\alpha = P_\alpha' Q_\alpha'$$

in which the recurring suffix implies, as usual, a summation of four terms or $P_\alpha Q_\alpha$ is invariant.

We call this the geometric product or scalar product of the two vectors, and it is frequently denoted by the symbol (**P** . **Q**).

If the scalar product is zero, we say that the vectors are perpendicular to one another.

Conversely, if any set of four quantities gives an invariant product when multiplied geometrically by an arbitrary four-vector, it must constitute a four-vector.

As may almost be anticipated without proof, all those groups of four quantities which we found in the earlier chapters to be subject to a simple Lorentz transformation constitute four-vectors, or rather they do so when the fourth member of each group is multiplied by ι.

Thus, if (x_1, x_2, x_3, x_4) are the space- (imaginary) time co-ordinates of a point-instant, and $x_1 + \delta x_1, \ldots, \ldots, x_4 + \delta x_4$ those of an adjacent point-instant,

$$\sum_\alpha \delta x_\alpha{}^2 = \sum_\alpha \delta x_\alpha{}'^2.$$

This invariant we write as δs^2; δs is, in fact, an element of the world-line of a particle which was situated at the first point at the first instant, and at the second point at the second instant. It is clear that $\delta s^2 = - \delta\tau^2$, $\delta\tau$ being the proper time of the earlier analysis. Now, δx_1, etc., transform just as x_1, etc., do, and constitute the components of a four-vector. Since δs is invariant, it follows that

$$dx_1/ds, \ldots, \ldots, dx_4/ds$$

are the components of a four-vector. They are evidently analogous to the direction cosines of an element of a curve in three dimensions.

Before proceeding to further examples of four-vectors, we can deal with another important invariant. By a well-known theorem in transformations

$$\delta x_1'\delta x_2'\delta x_3'\delta x_4' = J\, \delta x_1 \delta x_2 \delta x_3 \delta x_4$$

where J is the Jacobian of the transformation.* But this Jacobian is the determinant of the a coefficients, which, by Theorem I., is equal to unity. Hence

$$\delta x_1'\delta x_2'\delta x_3'\delta x_4' = \delta x_1 \delta x_2 \delta x_3 \delta x_4,$$

or an element of four-dimensional volume is invariant.

We are already familiar with this result in another guise.

* The Jacobian of the transformation is the determinant

$$\begin{vmatrix} \partial x_1'/\partial x_1, & \partial x_1'/\partial x_2, & \partial x'_1/\partial x_3, & \partial x_1'/\partial x_4 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \partial x_4'/\partial x_1, & \partial x_4'/\partial x_2, & \partial x_4'/\partial x_3, & \partial x_4'/\partial x_4 \end{vmatrix}$$

If v_x, v_y, v_z are the components of the velocity of a particle in a frame S, then

$$v_x = \delta x/\delta t = \iota \delta x_1/\delta x_4, \text{ etc.},$$

and therefore

$$v^2 = \Sigma v_x^2 = -(\delta x_1^2 + \delta x_2^2 + \delta x_3^2)/\delta x_4^2 = (\delta x_4^2 - \delta s^2)/\delta x_4^2.$$

Hence

$$dx_4/ds = (1 - v^2)^{-\frac{1}{2}} = \beta.$$

Consequently,

$$\delta x_4/\delta x_4' = \beta/\beta'$$

where v' is the velocity in a frame S′ and $\beta' = (1 - v'^2)^{-\frac{1}{2}}$.

Hence

$$\beta \delta x_1 \delta x_2 \delta x_3 = \beta' \delta x_1' \delta x_2' \delta x_3',$$

or an element of (three-dimensional) volume of a body as measured in a frame S varies as $1/\beta$ or $(1 - v^2)^{\frac{1}{2}}$, where v is the velocity of the body in the frame, a result which we deduced in Chapter II.

It follows at once that

$$\begin{aligned} dx_1/ds &= -\iota\beta v_x \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ dx_4/ds &= \beta \end{aligned}$$

and (multiplying by the factor ι), we see that βv_x, βv_y, βv_z, $\iota\beta$ are components of a four-vector, i.e.,

$$\beta' v_x' = a_{11}\beta v_x + a_{12}\beta v_y + a_{13}\beta v_z + a_{14}\iota\beta$$

and two similar equations, and

$$\iota\beta' = a_{41}\beta v_x + a_{42}\beta v_y + a_{43}\beta v_z + a_{44}\iota\beta.$$

But, after all, these transformation equations of the "velocity four-vector" can be written more succinctly thus:

$$dx_\lambda'/ds = a_{\lambda\alpha} dx_\alpha/ds,$$

in which is implied the whole of Einstein's kinematics.

In precisely the same fashion the transformation of accelerations is summed up in the equations

$$d^2x_\lambda/ds^2 = a_{\lambda\alpha} d^2x_\alpha/ds^2,$$

or in the statement that

$$\cdot \quad d^2x_1/ds^2, \ . \ . \ ., \ . \ . \ ., \ d^2x_4/ds^2$$

constitute a four-vector, the "acceleration four-vector." In terms of the three-dimensional velocity components of a particle's motion, these are

$$d(\beta v_x)/d\tau, \ . \ . \ ., \ . \ . \ ., \ \iota d\beta/d\tau$$

or

$$\beta d(\beta v_x)/dt, \ . \ . \ ., \ . \ . \ ., \ \iota\beta d\beta/dt.$$

Since

$$\Sigma(dx_a/ds)^2 = 1,$$

it follows that

$$dx_a/ds \ . \ d^2x_a/ds^2 = 0,$$

which we express by saying that the acceleration four-vector and the velocity four-vector are at right angles to one another. This is really a generalised form of the equation (4) of Chapter III.

Likewise the equations of motion and of energy of the third chapter are summarised in the equations

$$d(\mu dx_\lambda/ds)/ds = P_\lambda$$

where μ is an invariant and P_λ is a new four-vector, the "force four-vector," or, more strictly, the "force-activity" four-vector. For $P_1 \ . \ . \ . \ P_4$ are easily seen to be the $-\beta F_x$, . . ., . . ., $-\iota\beta(\mathbf{v} \, . \, \mathbf{F})$ of that chapter, so that the equations for transformation of force and activity are succinctly written:

$$P_\lambda' = a_{\lambda a}P_a.$$

Similarly, $\mu dx_\lambda/ds$ (after multiplication by ι) are the mv_x, . . ., . . ., ιm of Chapter III., and so constitute the "momentum-energy" four-vector.

Having rather hurriedly clothed the analysis of the earlier chapters in its vector symbolism, we should, to complete the analysis, demonstrate the four-vector property of the quantity $\mathbf{P}$ on the basis of the electro-magnetic equations. But in order to turn the analysis of Chapter IV. into vectorial form, we must extend our knowledge of vectorial mathematics somewhat further, so that we can not only achieve this purpose, but also use this powerful mathematical weapon to deal in the following chapter with the dynamics of continuous media and free ourselves from the narrow atmosphere of particle dynamics.

If $\phi(x_1, x_2, x_3, x_4)$ be a scalar function which transforms into $\psi(x_1', x_2', x_3', x_4')$ by the general Lorentz transformation (1) of Chapter V., we see that

$$\partial\psi/\partial x_1' = \partial\phi/\partial x_\alpha \,.\, \partial x_\alpha/\partial x_1'$$

and three similar equations.

By (6) of the last chapter,

$$\partial x_\alpha/\partial x_1' = a_{1\alpha}, \text{ etc.}$$

Hence

$$\partial\psi/\partial x_1' = a_{1\alpha}\partial\phi/\partial x_\alpha$$

and three similar equations ; or

$$\partial\psi/\partial x_\lambda' = a_{\lambda\alpha}\partial\phi/\partial x_\alpha.$$

Hence $\partial\phi/\partial x_\lambda$ constitute the components of a four-vector, which we call the Gradient of ϕ.

If P_λ is a four-vector which transforms into P_λ', it is not difficult by (2) and (3) of the last chapter to prove that

$$\partial P_\alpha/\partial x_\alpha = \partial P_\alpha'/\partial x_\alpha',$$

or $\partial P_\alpha/\partial x_\alpha$ is invariant.

By analogy with a similar result in three-dimensional analysis, we call $\partial P_\alpha/\partial x_\alpha$ the " Divergence " of the vector, and write it Div **P**, using the capital letter D to distinguish it from a three-dimensional divergence of a " three-vector," which we shall write " div."

Thus $\partial/\partial x_1, \ldots, \ldots, \partial/\partial x_4$ are the four components of a four-vector operator. Operating on a scalar function we obtain a vector function, the Gradient. A symbolic geometric multiplication of this operator and a vector yields an invariant scalar function, the Divergence.

TENSORS.

Let **P** and **Q** be two four-vectors, and let us write down in square array the following sixteen products :

$$\begin{array}{cccc} P_1Q_1, & P_1Q_2, & P_1Q_3, & P_1Q_4 \\ \cdot\quad\cdot & \cdot\quad\cdot & \cdot\quad\cdot & \\ \cdot\quad\cdot & \cdot\quad\cdot & \cdot\quad\cdot & \\ P_4Q_1, & P_4Q_2, & P_4Q_3, & P_4Q_4. \end{array}$$

If we transform to other orthogonal axes (the general Lorentz transformation), we can write down the sixteen corresponding quantities

$$P_\lambda'Q_\mu'$$

where P_λ' and Q_λ' are the components of the same vectors

along the new axes; it can be shown that each of the sixteen new quantities is a linear function of the original sixteen. In fact,

$$P_\lambda' Q_\mu' = a_{\lambda\alpha} a_{\mu\beta} P_\alpha Q_\beta,$$

which summarises sixteen equations with sixteen terms on the right-hand side of each (for we sum for each of the two dummy suffixes α and β).

Two properties of such a group of sixteen quantities are important.

1°. If $\mathbf{R}$ is an arbitrary four-vector, and the constituents of each row of the array are multiplied by R_1, R_2, R_3, R_4 respectively, and the four terms of each row added, we obtain four quantities, which are the components of a four-vector, i.e., are related with the transformed four by equations similar to (1) and (6) of Chapter V. Thus, in unaccented letters the quantities are

$$(Q_\alpha R_\alpha)\, P_1, \ldots, \ldots, (Q_\alpha R_\alpha)\, P_4.$$

With accented letters they are

$$(Q_\alpha' R_\alpha')\, P_1', \ldots, \ldots, (Q_\alpha' R_\alpha')\, P_4'.$$

But

$$Q_\alpha R_\alpha = Q_\alpha' R_\alpha' = (\mathbf{Q} \cdot \mathbf{R}).$$

Hence in each case we obtain the components of the vector $\mathbf{P}(\mathbf{Q} \cdot \mathbf{R})$.

A similar theorem is true if we substitute "column" for "row" in the enunciation, the vector in this case being

$$\mathbf{Q}(\mathbf{P} \cdot \mathbf{R}).$$

2°. If instead of multiplying by the components of a four-vector we operate in a similar fashion with the components $\partial/\partial x_\lambda$ of the vector operator, we obtain a four-vector; for since $\partial Q_\alpha/\partial x_\alpha = \partial Q_\alpha'/\partial x_\alpha' = \text{Div}\ \mathbf{Q}$ the results are the components

$$P_1(\partial Q_\alpha/\partial x_\alpha), \ldots, \ldots, P_4(\partial Q_\alpha/\partial x_\alpha)$$

of the four-vector $\mathbf{P}$ Div $\mathbf{Q}$.

Any set of sixteen quantities which are transformed like the sixteen quantities $P_\lambda Q_\mu$ constitute a "tensor of the second order." We denote such a tensor by double suffixed symbols, such as $P_{\lambda\mu}$, $Q_{\lambda\mu}$, $R_{\lambda\mu}$, etc., or by symbols such as $\mathfrak{P}$, $\mathfrak{Q}$, $\mathfrak{R}$. A four-vector P_λ or $\mathbf{P}$ is a tensor of the first order. The sixteen components of $P_{\lambda\mu}$ are denoted by

$$\begin{matrix} P_{11} & P_{12} & P_{13} & P_{14} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ P_{41} & P_{42} & P_{43} & P_{44} \end{matrix}$$

and the fundamental transformation equation for a tensor of the second order is

$$P_{\lambda\mu}' = a_{\lambda\alpha}a_{\mu\beta}P_{\alpha\beta}.$$

By means of this equation and equations (2)-(5) of the previous chapter, it is not difficult to prove these two theorems for any tensor of the second order, viz.,

1°. $P_{11}Q_1 + P_{12}Q_2 + P_{13}Q_3 + P_{14}Q_4$, and three similar expressions (or as we may succinctly write them, $P_{\lambda\alpha}Q_\alpha$), are components of a four-vector. We denote it by the symbol $[\mathbf{Q} \, . \, \mathfrak{P}]$.

2°. $\partial P_{\lambda\alpha}/\partial x_\alpha$ are components of a four-vector. Since each component of this vector *resembles* a divergence, this vector is sometimes called a "vector-divergence" and written Δiv $\mathfrak{P}$, the Greek Δ being used to distinguish it from the scalar divergences of three- and four-vectors, div and Div. It ought clearly to be understood, however, that any one of the components, say,

$$\partial P_{11}/\partial x_1 + \partial P_{12}/\partial x_2 + \partial P_{13}/\partial x_3 + \partial P_{14}/\partial x_4,$$

is not a scalar divergence, for P_{11}, P_{12}, P_{13}, P_{14} are not components of an *invariant* vector. P_{11}, P_{12}, P_{13}, P_{14} could, of course, be regarded as components of a vector, but it is not the same vector which has components P_{11}', P_{12}', P_{13}', P_{14}' with respect to the altered axes, because the relations between the $P_{\lambda\mu}'$ and $P_{\lambda\mu}$ are given above, and are *not*

$$P_{11}' = a_{11}P_{11} + \ . \ . \ . + a_{14}P_{14}, \text{ etc.}$$

To avoid any misapprehension of this sort, it is as well to use another name for the four-vector $\partial P_{\lambda\alpha}/\partial x_\alpha$, and it is frequently referred to, especially in English books, as the "Lorentzian of $\mathfrak{P}$," and written Lor $\mathfrak{P}$.

Symmetric and Anti-symmetric Tensors: Six-Vectors.

If a tensor of the second order has components which satisfy the relations

$$P_{\lambda\mu} = P_{\mu\lambda}$$

it is called "symmetric."

If its components satisfy the relations

$$P_{\lambda\mu} = - P_{\mu\lambda}$$

it is called " anti-symmetric." This involves the condition that $P_{11} = P_{22} = P_{33} = P_{44} = 0$, i.e., the " leading " components of an anti-symmetric tensor are all zero.

If **P** and **Q** are two vectors, then $P_\lambda Q_\mu - P_\mu Q_\lambda$ constitute the components of an anti-symmetric tensor.

An anti-symmetric tensor of the second order contains only six *numerically* different components, and selecting the six

$$P_{23},\ P_{31},\ P_{12},\ P_{14},\ P_{24},\ P_{34},$$

these are frequently referred to as the six components of a " six-vector." The reason for the name is based on the fact that if one takes (x_1, x_2, x_3, x_4) and (y_1, y_2, y_3, y_4) as the co-ordinates of a pair of points A and B in four dimensions, then

$$\begin{array}{lll} x_2y_3 - x_3y_2, & x_3y_1 - x_1y_3, & x_1y_2 - x_2y_1, \\ x_1y_4 - x_4y_1, & x_2y_4 - x_4y_2, & x_3y_4 - x_4y_3, \end{array}$$

are respectively double the areas of the projections of the triangle OAB on the *six* co-ordinate planes OX_2X_3, OX_3X_1, OX_1X_2, OX_1X_4, OX_2X_4, OX_3X_4. In three-dimensional geometry corresponding expressions (three in number) are known to constitute the components of an invariant three-vector (" axial " vector), whose magnitude is double the area of the triangle OAB, and whose direction is normal to its plane OAB. By analogy the six projections above are referred to as components of a six-vector, whose magnitude is double the area of the triangle OAB, and whose " orientation " is the plane of the triangle. (As in four dimensions there is an infinity of lines perpendicular to a plane at any point of that plane, and as all these lines lie in another plane intersecting the former plane in the *one* point only, we cannot speak of the " direction " of a six-vector.) In general, P and Q being four-vectors, $P_2Q_3 - P_3Q_2, \ldots, \ldots, \ldots, \ldots, P_3Q_4 - P_4Q_3$ are the six components of a six-vector; and since $P_\lambda Q_\mu - P_\mu Q_\lambda$ is an anti-symmetric tensor, we can also say that if $\mathfrak{P}$ is any *anti-symmetric* tensor, six of its components properly selected constitute a six-vector, P_{23} being the component of the six-vector in the plane OX_2X_3, etc. As a matter of fact, the sixteen equations of transformation for a tensor of the second order, with sixteen terms on the right-hand side of each, reduce, in the case of an anti-symmetric tensor to six equations with six terms on the right-hand side of each. For example,

$$P_{\lambda\lambda}' = a_{\lambda\alpha}a_{\lambda\beta}P_{\alpha\beta}$$
$$= 0,$$

since

$$P_{\alpha\beta} = -P_{\beta\alpha} \text{ and } P_{\alpha\alpha} = 0.$$

Also

$$P_{\lambda\mu}' = a_{\lambda\alpha}a_{\mu\beta}P_{\alpha\beta} = -a_{\mu\beta}a_{\lambda\alpha}P_{\beta\alpha} = -P_{\mu\lambda}'.$$

So the anti-symmetry is preserved after transformation, and we can write those equations of transformation, which are not either zero identities or duplicates of others, in the form

$$P_{\lambda\mu}' = \begin{vmatrix}\lambda\mu\\23\end{vmatrix}P_{23} + \begin{vmatrix}\lambda\mu\\31\end{vmatrix}P_{31} + \begin{vmatrix}\lambda\mu\\12\end{vmatrix}P_{12} + \begin{vmatrix}\lambda\mu\\14\end{vmatrix}P_{14} + \begin{vmatrix}\lambda\mu\\24\end{vmatrix}P_{24} + \begin{vmatrix}\lambda\mu\\34\end{vmatrix}P_{34}$$

where $\begin{vmatrix}\lambda\mu\\\alpha\beta\end{vmatrix}$ is a contraction for $a_{\lambda\alpha}a_{\mu\beta} - a_{\mu\alpha}a_{\lambda\beta}$ and $\lambda\mu$ is replaced consecutively by 23, 31, 12, 14, 24, 34 in order to give the six equations.

It will be convenient for the moment to write the thirty-six coefficients $\begin{vmatrix}\lambda\mu\\\alpha\beta\end{vmatrix}$ above as $b_{11}, b_{12}, \ldots, b_{16}, b_{21}, \ldots, b_{26}, \ldots, b_{66}$, so that, for example,

$$b_{11} = \begin{vmatrix}23\\23\end{vmatrix},\ b_{16} = \begin{vmatrix}23\\34\end{vmatrix},\ b_{21} = \begin{vmatrix}31\\23\end{vmatrix},\ b_{24} = \begin{vmatrix}31\\14\end{vmatrix},\ b_{53} = \begin{vmatrix}24\\12\end{vmatrix}, \text{ etc.}$$

The equations just written are then

$$\mathbb{P}_\lambda' = b_{\lambda 1}\mathbb{P}_1 + b_{\lambda 2}\mathbb{P}_2 + b_{\lambda 3}\mathbb{P}_3 + b_{\lambda 4}\mathbb{P}_4 + b_{\lambda 5}\mathbb{P}_5 + b_{\lambda 6}\mathbb{P}_6 \quad (10\text{A})$$
$$= b_{\lambda\alpha}\mathbb{P}_\alpha \text{ (summing for } \alpha \text{ from 1 to 6).}$$

By Theorem III., of Chapter V., concerning the second minors of the determinant of the a-coefficients, it is easily seen that if we write the 36 b-coefficients in a similar square array, the constituent in row r and column s is equal to the constituent in the row $(r + 3)$, and column $(s + 3)$ (where r and s can be replaced individually by 1, 2, or 3). Making use of this, we can see that the transformation equations for a six-vector are equivalent to

$$\mathbb{P}_4' = b_{11}\mathbb{P}_4 + b_{12}\mathbb{P}_5 + b_{13}\mathbb{P}_6 + b_{14}\mathbb{P}_1 + b_{15}\mathbb{P}_2 + b_{16}\mathbb{P}_3$$
$$\mathbb{P}_5' = b_{21}\mathbb{P}_4 + b_{22}\mathbb{P}_5 + b_{23}\mathbb{P}_6 + b_{24}\mathbb{P}_1 + b_{25}\mathbb{P}_2 + b_{26}\mathbb{P}_3$$
$$\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots$$
$$\mathbb{P}_3' = b_{61}\mathbb{P}_4 + b_{62}\mathbb{P}_5 + b_{63}\mathbb{P}_6 + b_{64}\mathbb{P}_1 + b_{62}\mathbb{P}_2 + b_{63}\mathbb{P}_3.$$

Which proves the proposition that if

$$P_{23}, P_{31}, P_{12}, P_{14}, P_{24}, P_{34}$$

are the components of a six-vector in the planes OX_2X_3, . . ., OX_3X_4, respectively, then

$$P_{14}, P_{24}, P_{34}, P_{23}, P_{31}, P_{12}$$

are the components in the planes OX_2X_3, . . ., OX_3X_4 respectively of another six-vector.

The second six-vector is said to be "reciprocal to," or "associated with," the first.

The six-vector $P_\lambda Q_\mu - P_\mu Q_\lambda$ is called the vector product of the two four-vectors $\mathbf{P}$ and $\mathbf{Q}$ and is generally represented symbolically by $[\mathbf{P} . \mathbf{Q}]$.

In particular, the six quantities

$$\partial P_\mu/\partial x_\lambda - \partial P_\lambda/\partial x_\mu,$$

which constitute the symbolic vector product of the four-vector operator $\partial/\partial x_\lambda$ and the four-vector P_λ, form a six-vector. It is called the Curl or Rotation of $\mathbf{P}$, and we write it

$$\text{Curl } \mathbf{P} \text{ or Rot } \mathbf{P},$$

capital C or R being used to prevent confusion with the well-known three-dimensional curl or rotation. (The symbol Rot is frequently used in continental works. English authors generally prefer Curl, following Heaviside's example.)

The square array of the thirty-six b-coefficients above can be shown to have properties similar to some of those possessed by the array of the sixteen a-coefficients.

Thus

$$\begin{aligned} b_{\lambda a} b_{\mu a} &= 1 \text{ if } \lambda = \mu \\ &= 0 \text{ if } \lambda \neq \mu \\ b_{a\lambda} b_{a\mu} &= 1 \text{ if } \lambda = \mu \\ &= 0 \text{ if } \lambda \neq \mu \end{aligned}$$

(the summation for a being from 1 to 6).

By reason of these we can prove that

$$\mathfrak{P}_a' \mathfrak{P}_a' = \mathfrak{P}_a \mathfrak{P}_a.$$

This invariant is therefore the square of the magnitude of the six-vector $\mathfrak{P}$.

If $\mathfrak{P}$ and $\mathfrak{R}$ are two six-vectors, it can be shown likewise that

$$\mathfrak{P}_a \mathfrak{R}_a = \mathfrak{P}_a' \mathfrak{R}_a',$$

i.e., $\mathfrak{P}_a \mathfrak{R}_a$ is invariant. The sum of these six terms is referred

to as the "scalar" or "geometric" product of $\mathfrak{P}$ and $\mathfrak{R}$, and can be denoted by $(\mathfrak{P} \,.\, \mathfrak{R})$.

One more result is required for the purposes of Relativity. It refers to a tensor derived from a six-vector $\mathfrak{F}$, as follows: Let $F_{\lambda\mu}$ be a component of $\mathfrak{F}$, and $R_{\lambda\mu}$ be the corresponding component of the reciprocal six-vector, so that $R_{23} = F_{14}$, . . ., $R_{14} = F_{23}$. The tensor in question is

$$T_{\lambda\mu} = F_{\lambda\alpha}F_{\mu\alpha} - R_{\lambda\alpha}R_{\mu\alpha} \text{ (the summation for } \alpha \text{ is from 1 to 4).}$$

That $T_{\lambda\mu}$ is a tensor can be proved most readily by working out the values for the four quantities $\partial T_{\lambda\beta}/\partial x_\beta$. It will be found that

$$\begin{aligned} \partial T_{\lambda\beta}/\partial x_\beta &= F_{\lambda\alpha}\partial F_{\beta\alpha}/\partial x_\beta - R_{\lambda\alpha}\partial R_{\beta\alpha}/\partial x_\beta \\ &= F_{\lambda\alpha}P_\alpha - R_{\lambda\alpha}\, Q_\alpha, \end{aligned}$$

where $\mathbf{P}$ and $\mathbf{Q}$ are the four-vectors Lor $\mathfrak{F}$ and Lor $\mathfrak{R}$ respectively.

Since $F_{\lambda\alpha}\, P_\alpha$ and $R_{\lambda\alpha}\, Q_\alpha$ are the components of the four-vector products $[\mathbf{P} \,.\, \mathfrak{F}]$ and $[\mathbf{Q} \,.\, \mathfrak{R}]$, it follows that $\partial T_{\lambda\beta}/\partial x_\beta$ are components of a four-vector. We know that if $U_{\lambda\mu}$ is a tensor of the second order, then $\partial U_{\lambda\alpha}/\partial x_\alpha$ is a four-vector; and the converse is equally true and not difficult to prove. Hence $T_{\lambda\mu}$ is a tensor of the second order. It is a symmetric tensor; moreover, it is a special type of symmetric tensor, for on examination it will be found that the sum of its leading components,

$$T_{11} + T_{22} + T_{33} + T_{44},$$

is zero.

Having dealt with the vectorial material which is necessary for our future progress, we can recapitulate what that material is. It consists in the main of the following nine functions and their properties:

(1) The invariant scalar product of two four-vectors, $(\mathbf{P} \,.\, \mathbf{Q})$.

(2) The invariant scalar product of two six-vectors, $(\mathfrak{P} \,.\, \mathfrak{Q})$.

(3) The six-vector product of two four-vectors, $[\mathbf{P} \,.\, \mathbf{Q}]$.

(4) The four-vector product of a four-vector and a tensor of the second order, and in particular of a four-vector and a six-vector, $[\mathbf{P} \,.\, \mathfrak{P}]$.

(5) The tensor derived from a six-vector and its reciprocal.

(6) The four-vector Gradient of a scalar function.

(7) The scalar Divergence of a four-vector function.

(8) The six-vector Curl or Rotation of a four-vector function.

(9) The four-vector Lorentzian of a tensor of the second order, and in particular of a six-vector.

Equations of the Electromagnetic Field.

The essential feature of these vectors and tensors is the existence of linear relations (with constant coefficients) which connect the transformed components with the originals. Hence, if the laws embodying physical results in a definite frame can be expressed in equations between such tensor functions, the equalities are preserved after a general Lorentz transformation, i.e., the laws are true for physical results in any other frame moving with a uniform velocity relative to the first. This tensor analysis is the natural mathematical medium for expressing laws of nature in the invariant form required by the first postulate of Relativity, and we have seen how it can be employed in the treatment of kinematics and the laws of motion. We now pass on to the treatment of the electromagnetic field and other problems arising out of it.

In order to avoid confusion in our symbolism between three-vectors and four-vectors, we will keep small Clarendon and italic type for the former, and capital Clarendon and italic type for the latter. Thus the electric vector will be denoted by $\mathbf{e}$, components e_x, e_y, e_z; the magnetic vector by $\mathbf{h}$, components h_x, h_y, h_z; velocity of electrified matter at a point by $\mathbf{v}$, components v_x, v_y, v_z; and electric density by ρ.

The equations of the field are

$$\operatorname{curl} \mathbf{h} - \partial \mathbf{e}/\partial t = \rho \mathbf{v} \qquad (1)$$

$$\operatorname{div} \mathbf{e} = \rho \qquad (2)$$

$$\operatorname{curl} \mathbf{e} + \partial \mathbf{h}/\partial t = 0 \qquad (3)$$

$$\operatorname{div} \mathbf{h} = 0 \qquad (4)$$

or, in Cartesian form,

$$\partial h_z/\partial y - \partial h_y/\partial z - \partial e_x/\partial t = \rho v_x \qquad (1\text{A})$$

and two similar equations,

$$\partial e_x/\partial x + \partial e_y/\partial y + \partial e_z/\partial z = 0 \qquad (2\text{A})$$

$$\partial e_z/\partial y - \partial e_y/\partial z + \partial h_x/\partial t = 0 \qquad (3\text{A})$$

and two similar equations,

$$\partial h_x/\partial x + \partial h_y/\partial y + \partial h_z/\partial z = 0 \qquad (4\text{A})$$

Let us write x_1, x_2, x_3, x_4 for $x, y, z, \iota t$;
F_{23} for h_x, F_{31} for h_y, F_{12} for h_z,
F_{14} for $-\iota e_x$, F_{24} for $-\iota e_y$, F_{34} for $-\iota e_z$.

A little trouble will convince one that equations (1) and (2) become the symmetrical group of four:

$$\partial F_{\lambda 1}/\partial x_1 + \partial F_{\lambda 2}/\partial x_2 + \partial F_{\lambda 3}/\partial x_3 + \partial F_{\lambda 4}/\partial x_4 = \rho v_\lambda, \quad (5)$$

where λ is made 1, 2, 3, 4 in succession and $(v_1, v_2, v_3, v_4) = (v_x, v_y, v_z, \iota)$, provided we make

$$F_{11} = F_{22} = F_{33} = F_{44} = 0,$$

and assume

$$F_{\lambda\mu} = - F_{\mu\lambda}.$$

If the four expressions on the left-hand side were the components of a four-vector, and likewise the expressions on the right-hand side, the principle of Relativity would be satisfied as far as equations (1) and (2) are concerned; for the equality of two four-vectors is independent of the particular axes chosen in the World, i.e., of the particular frame chosen among the group of frames moving uniformly relative to one another in space. The first proviso is satisfied if $F_{\lambda\mu}$ is an anti-symmetric tensor or six-vector, for the left-hand side would then express the four components of its Lorentzian. So we could write these equations (5) succinctly in the forms

$$\partial F_{\lambda\alpha}/\partial x_\alpha = J_\lambda \quad . \quad . \quad . \quad (5\text{A})$$

or

$$\text{Lor } \mathfrak{F} = \mathbf{J}, \quad . \quad . \quad . \quad (5\text{B})$$

where J_1, J_2, J_3, J_4 are the components of a four-vector, the "stream" four-vector, and equal respectively to ρv_x, ρv_y, ρv_z, $\iota\rho$. $\mathfrak{F}$ is usually called the "field" six-vector.

Of course, the assumption that $\mathfrak{F}$ is a six-vector (which is necessary, if the equations are to pass the restricted relativity test) means that if the values of $\mathbf{h}$ and $\mathbf{e}$ measured in another frame S′ are denoted by accented letters, then these six equations are true:

$$F_{\lambda\mu}' = \begin{vmatrix} \lambda\mu \\ \alpha\beta \end{vmatrix} F_{\alpha\beta} \quad . \quad . \quad . \quad . \quad (6)$$

($\alpha\beta$ going through the sequence 23, 31, 12, 14, 24, 34 in the summation). This implies that any component of the electric or magnetic fields in S′ is a linear function of all the six components of the field in S, the coefficients being certain constants depending on the relative motion of the frames and the relative orientation of the axes in each frame. In Chapter IV. we obtained these relations for the simple Lorentz transformation; in that case, certain of the coefficients were zero, so the right-hand side did not contain the full number of six terms.

Similarly, the assumption that $\mathbf{J}$ is a four-vector means that the components of the three-vectorial current density and

the scalar charge density, as measured in a frame S′, are connected by definite linear relations with the similar four quantities as measured in S. Thus

$$\left.\begin{array}{ll} & \rho' v_x' = a_{11}\rho v_x + a_{12}\rho v_y + a_{13}\rho v_z + a_{14}\,\iota\rho \\ \text{and two similar equations} & \\ \text{and} & \iota\rho' = a_{41}\rho v_x + a_{12}\rho v_y + a_{43}\rho v_z + a_{44}\iota\rho \end{array}\right\} \quad . \quad (7)$$

The relations obtained in Chapter IV. were the degenerate form of these for the simple Lorentz transformation.

As a matter of fact, it is not difficult to throw equations (7), (8), and (9) of Chapter IV. into a three-vectorial form which is equivalent to (6) and (7) above. Thus equations (7) and (8) of Chapter IV. become (after making the change from large letters to small):

$$\left.\begin{array}{l} \mathbf{e}' = \alpha\{\mathbf{e} + [\mathbf{u}\,.\,\mathbf{h}]\} - (\alpha - 1)(\mathbf{u}\,.\,\mathbf{e})\mathbf{u}/u^2 \\ \mathbf{h}' = \alpha\{\mathbf{h} - [\mathbf{u}\,.\,\mathbf{e}]\} - (\alpha - 1)(\mathbf{u}\,.\,\mathbf{h})\mathbf{u}/u^2 \end{array}\right\} \quad . \quad (6\text{A})$$

Equations (9) of Chapter V. become:

$$\left.\begin{array}{l} \rho'\mathbf{v}' = \rho\mathbf{v} - \alpha\rho\mathbf{u} + (\alpha - 1)\rho(\mathbf{u}\,.\,\mathbf{v})\mathbf{u}/u^2 \\ \rho' = \alpha\{\rho - \rho(\mathbf{u}\,.\,\mathbf{v})\} \end{array}\right\} \quad . \quad (7\text{A})$$

(Compare equations (E) and (F) of Chapter V.)

We have still to deal with the equations (3) and (4). It will be found that they can be written as four equations in the form,

$$\partial F_{\mu\nu}/\partial x_\lambda + \partial F_{\nu\lambda}/\partial x_\mu + \partial F_{\lambda\mu}/\partial x_\nu = 0, \quad . \quad . \quad (8)$$

where we write for λ, μ, ν:

2, 3, 4 in the case of the first equation ;
1, 3, 4 „ „ „ second „
1, 2, 4 „ „ „ third „
1, 2, 3 „ „ „ fourth „

But for our immediate purpose, if we consider the anti-symmetric tensor reciprocal to $F_{\lambda\mu}$, viz., $R_{\lambda\mu}$, where $R_{23} = F_{14}$, etc.; $R_{11} = 0$, etc., we can write (6) in the form:

$$\partial \mathbf{R}_{\lambda\alpha}/\partial x_\alpha = 0 \quad . \quad . \quad . \quad (8\text{A})$$

or, using the six-vector reciprocal to $\mathfrak{F}$, in the form

$$\text{Lor}\ \mathfrak{R} = 0 \quad . \quad . \quad . \quad . \quad . \quad (8\text{B})$$

Thus the equations (3) and (4), being expressible in vectorial form, on the same assumptions as before as to the relations between the field components in the frames S and S′, also pass

the Relativity test. In fine, the conditions for the relativity of the field equations are that $\mathfrak{F}$, i.e. ($\mathbf{h}$, $-\iota\mathbf{e}$) is a six-vector, and $\mathbf{J}$, i.e. ($\rho\mathbf{v}$, $\iota\rho$) is a four-vector. That is the succinct way of stating the equations (6) and (7) above.

Electromagnetic Potential.

If we take any four-vector function $\mathbf{A}$ and obtain its Curl, i.e., $\partial A_3/\partial x_2 - \partial A_2/\partial x_3$, and five similar components, the result is a six-vector. To any six-vector we can apply the " Lorentzian " operator and obtain a four-vector. Now, although it is not in general true that the Lorentzian of Curl $\mathbf{A}$ is zero, it will be found on trial that the Lorentzian of the six-vector reciprocal to Curl $\mathbf{A}$ is zero. Now, since Lor $\mathfrak{R}$ is zero (equation (8B)), where $\mathfrak{R}$ is reciprocal to $\mathfrak{F}$, the field vector, this suggests that a four-vector $\mathbf{A}$ exists such that

$$\mathfrak{F} = \text{Curl } \mathbf{A} \quad . \quad . \quad . \quad . \quad (9)$$

If this is so, then by (5),

$$\text{Lor Curl } \mathbf{A} = \mathbf{J}. \quad . \quad . \quad . \quad (10)$$

It will appear on examination that equation (9) is the six-vectorial form of the six three-vector equations :

$$\mathbf{h} = \text{curl } \mathbf{a}$$
$$\mathbf{e} = -\text{ grad } \phi - \partial\mathbf{a}/\partial t,$$

where $A_1 = a_x$, $A_2 = a_y$, $A_3 = a_z$, $A_4 = \iota\phi$.

So we have been really giving its general vectorial form to the results obtained as equations (10)-(12) of Chapter IV., where we demonstrated the " cogrediency " of the three components of the vector potential and the scalar potential, i.e., establishing the fact that the three components of the vector potential and the scalar potential (affected by ι) constitute a four-vector. (Note that in Chapter IV. we were using the symbol $\mathbf{A}$ for the three-vector potential of the field, whereas we are now using $\mathbf{a}$ for that function, keeping $\mathbf{A}$ for the four-vector " electromagnetic potential.")

Equation (10) is a succinct four-vectorial form of two very well-known results concerning the vector and scalar potentials. Thus the first of the Cartesian equations in terms of space and time co-ordinates in the frame S, obtained by translating (10) into the more customary symbols, turns out to be

$$\partial(\text{div } \mathbf{a} + \partial\phi/\partial t)/\partial x - \Box a_x = \rho v_x,$$

where $\square$ is the "Dalembertian" operator

$$\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2 - \partial^2/\partial t^2.$$

In fact, the first three equations obtained by translation of (10) are summarised in the three-vectorial form

$$\text{grad}\,(\text{div}\,\mathbf{a} + \partial\phi/\partial t) - \square\mathbf{a} = \rho\mathbf{v}.$$

The fourth of (10) turns out to be

$$\Delta\phi + \partial(\text{div}\,\mathbf{a})/\partial t + \rho = 0,$$

where Δ is the "Laplacian" operator

$$\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2.$$

Hence if, as is generally assumed in electromagnetic theory, we subject the vector potential $\mathbf{a}$ to the restriction

$$\text{div}\,\mathbf{a} + \partial\phi/\partial t = 0,$$

we have proved that

$$\square\mathbf{a} + \rho\mathbf{v} = 0$$

and

$$\square\phi + \rho \quad = 0,$$

or, in four-vectorial form, the four-vector $\mathbf{A}$ satisfies the condition

$$\square\mathbf{A} + \mathbf{J} = 0, \quad . \quad . \quad . \quad . \quad (11)$$

where $\square$ is still the "Dalembertian" operator but is now the symbol representing the operation

$$\partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + \partial^2/\partial x_3^2 + \partial^2/\partial x_4^2,$$

being, in fact, the invariant "square" of the four-vector operator $\partial/\partial x_\lambda$.

Equations (9), (10), (11) are interesting, not merely on account of the elegant brevity with which they express well-known equations of electromagnetic theory, but because, being in vectorial form, they demonstrate the relativity of these equations.

Conservation of Charge.

It can easily be seen that the Scalar Divergence of the Lorentzian of any six-vector is identically zero, as the twelve terms cancel in pairs Hence

$$\text{Div (Lor } \mathfrak{F}) = 0,$$

and so by (5)

$$\text{Div } \mathbf{J} = 0.$$

In Cartesian form this is

$$\partial(\rho v_x)/\partial x + \partial(\rho v_y)/\partial y + \partial(\rho v_z)/\partial z + \partial\rho/\partial t = 0,$$

or

$$\text{div } (\rho\mathbf{v}) + \dot{\rho} = 0,$$

which is the mathematical expression of the fact that any increase or decrease of electric charge within a given region definitely marked out in the frame S is entirely accounted for by a flow inwards or outwards across the bounding surface, there being no creation or destruction of charge in the region. So conservation of charge is likewise an invariant relation, i.e., if true in one frame it is true in all.

Invariant Quantities.

Since $\mathbf{J}$ is a four-vector, its magnitude is invariant; this is the square root of $\rho^2(v_x{}^2 + v_y{}^2 + v_z{}^2 - 1)$, or (ignoring ι) $\rho(1 - v^2)^{\frac{1}{2}}$. As we saw in Chapter IV., this means that the charge on a definite body is estimated alike by observers in any frame.

Also, the magnitude of $\mathfrak{F}$ is invariant; this is

$$(\Sigma h_x{}^2 - \Sigma e_x{}^2)^{\frac{1}{2}};$$

or $h^2 - e^2$ is the same for all observers.

The scalar product of two six-vectors is invariant, therefore $(\mathfrak{F} . \mathfrak{R})$ is invariant. On examination, this turns out to be the invariance of the geometric product of the electric and magnetic three-vectors, viz., $(\mathbf{h} . \mathbf{e})$.

Relativity of Electromagnetic Force and its Activity.

We have seen that we can form a four-vector product of a four-vector and a six-vector. Let us form it for the stream vector $\mathbf{J}$ and the field vector $\mathfrak{F}$. The components of $[\mathbf{J} . \mathfrak{F}]$ are $J_1F_{11} + J_2F_{12} + J_3F_{13} + J_4F_{14}$, and three similar expressions. On examination the first reveals itself in Cartesian form as $\rho(e_x + v_yh_z - v_zh_y)$; and the next two complete the triad of components parallel to the axes in the frame S of the electromagnetic force *on unit volume* of a charged body moving relative

to the frame with a velocity $\mathbf{v}$, viz., $\rho(\mathbf{e} + [\mathbf{v} \,.\, \mathbf{h}])$. Let us call this $\rho\mathbf{f}$, where $\mathbf{f}$ is the force *on unit charge*. The fourth component of $[\mathbf{J} \,.\, \mathfrak{F}]$ is found to be $\iota\rho(v_x e_x + v_y e_y + v_z e_z)$, which is identically equal to $\iota\rho(v_x f_x + v_y f_y + v_z f_z)$, since terms involving h_x, h_y, h_z in the latter cancel in pairs. This fourth component is, therefore (ignoring ι), the activity of the force per unit volume $(\mathbf{v} \,.\, \mathbf{f})$ being the activity of the force per unit charge. Now it has been established above that ρ/β is invariant (where $\beta = 1/\sqrt{(1 - v^2)}$. Hence it follows that if $\mathbf{f}$ is the force *per unit charge* on a body βf_x, βf_y, βf_z, $\iota\beta(\mathbf{v} \,.\, \mathbf{f})$ constitute the components of a four-vector, and since the charge on a body has been proved to be invariant, it appears that the electromagnetic force-components on a given charged body, and the force activity (affected by ι) when each is multiplied by β constitute the components of a four-vector. We have seen a few pages back that this is the condition required to establish the relativity of the equations of motion.

So far the interest in this four-dimensional vectorial analysis has lain in the elegant and succinct form it has given to the expression of a number of results concerning the relativity of equations which we already had discovered in connection with the simple Lorentz transformation. As a matter of fact, it is a mathematical method as powerful as it is elegant, and it is time to employ it in developing other results not dealt with so far, and in showing how by natural steps we are led into the treatment of the dynamics of a continuous medium.

It was pointed out above that if $\mathfrak{F}$ be a six-vector, then we can form a tensor of the second order with components $F_{\lambda a}F_{\mu a} - R_{\lambda a}R_{\mu a}$, where $\mathfrak{R}$ is reciprocal to $\mathfrak{F}$.

Take $\mathfrak{F}$ to represent the field tensor and form the tensor $\frac{1}{2}(R_{\lambda a}R_{\mu a} - F_{\lambda a}F_{\mu a})$. To facilitate this, let us write down $\mathfrak{F}$ and $\mathfrak{R}$ in full as anti-symmetric tensors:

$$
\begin{array}{lcccc}
F_{\lambda\mu} \text{ is} & 0 & h_z & -h_y & -\iota e_x \\
 & -h_z & 0 & h_x & -\iota e_y \\
 & h_y & -h_x & 0 & -\iota e_z \\
 & \iota e_x & \iota e_y & \iota e_z & 0 \\
R_{\lambda\mu} \text{ is} & 0 & -\iota e_z & \iota e_y & h_x \\
 & \iota e_z & 0 & -\iota e_x & h_y \\
 & -\iota e_y & \iota e_x & 0 & h_z \\
 & -h_x & -h_y & -h_z & 0
\end{array}
$$

On working out $\frac{1}{2}(R_{\lambda a}R_{\mu a} - F_{\lambda a}F_{\mu a})$ for all possible values of λ and μ, we obtain the tensor $E_{\lambda\mu}$. Its constituents are

$$
\begin{aligned}
E_{11} &= \tfrac{1}{2}(e_x^2 - e_y^2 - e_z^2 + h_x^2 - h_y^2 - h_z^2) \\
E_{22} &= \tfrac{1}{2}(e_y^2 - e_z^2 - e_x^2 + h_y^2 - h_z^2 - h_x^2) \\
E_{33} &= \tfrac{1}{2}(e_z^2 - e_x^2 - e_y^2 + h_z^2 - h_x^2 - h_y^2) \\
E_{44} &= \tfrac{1}{2}(e^2 + h^2) \\
E_{23} &= e_y e_z + h_y h_z = E_{32} \\
E_{31} &= e_x e_z + h_x h_z = E_{13} \\
E_{12} &= e_x e_y + h_x h_y = E_{21} \\
E_{14} &= -\iota(e_y h_z - e_z h_y) = E_{41} \\
E_{24} &= -\iota(e_z h_x - e_x h_z) = E_{42} \\
E_{34} &= -\iota(e_x h_y - e_y h_x) = E_{43}.
\end{aligned}
$$

On examination it will be seen that we have written down expressions for well-known concepts of the electromagnetic field. Thus

$$
\begin{matrix}
E_{11} & E_{12} & E_{13} \\
E_{21} & E_{22} & E_{23} \\
E_{31} & E_{32} & E_{33}
\end{matrix}
$$

are the nine components of the *three-dimensional* stress tensor of Maxwell; E_{14}, E_{24}, E_{34}, on omission of the factor $-\iota$, are the densities of the components of the "electromagnetic momentum" of the field, viz., $[\mathbf{e} \cdot \mathbf{h}]$; and E_{44} is the energy density of the field.*

With reference to a frame S, we can refer to the Maxwell (symmetrical) stress tensor as having the components

$$
\begin{matrix}
t_{xx} & t_{xy} & t_{xz} \\
t_{yx} & t_{yy} & t_{yz} \\
t_{zx} & t_{zy} & t_{zz}
\end{matrix}
$$

where $t_{xx} = E_{11}$, etc.

Also, we shall denote the momentum-density by $\mathbf{g}$, so that $\mathbf{g} = [\mathbf{e} \cdot \mathbf{h}]$; $g_x = e_y h_z - e_z h_y = \iota\, E_{14}$, etc. The energy density we denote by ϵ, so that ϵ is E_{44}. The energy-stream vector of Poynting we represent by $\mathbf{s}$, so that $\mathbf{s}$ is also $[\mathbf{e} \cdot \mathbf{h}]$; $s_x = \iota\, E_{41}$.†

Now, it is well known that in electromagnetic theory we have the following four equations connecting these quantities with the electromagnetic force and activity dealt with above,

$$
\left.
\begin{aligned}
&\rho f_x = \partial t_{xx}/\partial x + \partial t_{xy}/\partial y + \partial t_{xz}/\partial z - \partial g_x/\partial t \\
&\text{and two similar equations,} \\
&\text{and } -\rho(\mathbf{v} \cdot \mathbf{f}) = \partial s_x/\partial x + \partial s_y/\partial y + \partial s_z/\partial z + \partial \epsilon/\partial t
\end{aligned}
\right\} \quad . \quad (12)
$$

* We are using Lorentz units; hence the non-appearance of the familiar factor 4π. Also we are using the relativity unit of time; hence the non-appearance of c.

† This apparent redundancy of symbols will be explained presently.

These are mathematical equations which are true in virtue of the assumed truth of the equations of the field (1), (2), (3), (4) in the frame. Their particular form brings to the mind of the mathematical physicist who is familiar with them the idea of "conservation," e.g., any of the first three can be written

$$\partial g_\lambda/\partial t = -\rho f_\lambda + \partial t_{\lambda x}/\partial x + \partial t_{\lambda y}/\partial y + \partial t_{\lambda z}/\partial z$$

(where λ is replaced by 1, 2, 3 in succession), and these can be interpreted as follows. In the frame S there is a continuous medium possessing a momentum $\mathbf{g}$ per unit of volume. It is in a state of strain which is responsible for a stress whose components are t_{xx}, t_{xy}, etc. Then $-\rho\mathbf{f}$ is the magnitude of the "body force" on it per unit of volume. For, as is well known, the equation then expresses the law that the increase of momentum per unit time within a surface surrounding a definite portion of the medium is equal to the sum of the forces on that part, due to the stresses exerted across the surface by the external part, and the body forces. In the days of the "stagnant ether" theory this was a most natural interpretation in view of the prevalent desire to reduce the explanation of all phenomena to the laws of motion; it was the truth of these equations and this natural desire which led to the introduction of purely mechanical terms, so that $\mathbf{g}$, or $[\mathbf{e}\cdot\mathbf{h}]$ was called "electromagnetic momentum;" t_{xx}, etc., was called "stress in the ether," and $-\rho\mathbf{f}$ was called "force on the ether;" for in this latter case, as $\rho\mathbf{f}$ was at all events force on electrified matter (supposed to be transmitted by the ether from other electrified matter), the opposite of this, viz., $-\rho\mathbf{f}$, was regarded as the reactionary force of electrified matter on ether—an implicit appeal to the third law of motion. One difficulty, viz., that a strictly *stagnant* ether could hardly be said to possess momentum in the ordinary mechanical sense, was evaded by the assumption that the ether must relatively to ordinary matter be possessed of tremendous density (ideas such as this were very highly developed by Lodge), and so the necessary amounts of "momentum," or $[\mathbf{e}\cdot\mathbf{h}]$, per unit volume in any known fields could be easily allowed to exist with such minute velocities for the ether as to leave it "practically immobile." Similarly, the fourth equation of (12), being written as

$$\partial\epsilon/\partial t = -\rho(\mathbf{v}\cdot\mathbf{f}) - \operatorname{div}\mathbf{s},$$

was interpreted as a "conservation of energy" law; this led

to the definition of ϵ or $\frac{1}{2}(e^2 + h^2)$ as the "electromagnetic energy" per unit volume, and so the increase of energy within a given portion of the ethereal medium was regarded as being supplied by the activity $-\rho(\mathbf{v}\cdot\mathbf{f})$ of the body force $-\rho\mathbf{f}$ per unit volume, together with a flow across the bounding surface of energy at the rate $\mathbf{s}$ per unit area per unit time. There is one matter which requires notice before passing on. The symbol $[\mathbf{e}\cdot\mathbf{h}]$ has appeared both as density of "momentum" and as "energy-stream" density; this apparent contradiction of the facts of dimensions arises from our special unit of time. As stated before, to return to usual units, we must replace $\mathbf{v}$ by $\mathbf{v}/c$ and t by ct, so that (12) would appear as

$$\rho f_x = \Sigma \partial t_{xx}/\partial x - c^{-1}\partial(e_y h_z - e_z h_y)/\partial t$$

and two similar equations,

and $$-\rho(\mathbf{v}\cdot\mathbf{f}) = c \operatorname{div}[\mathbf{e}\cdot\mathbf{h}] + \partial\epsilon/\partial t,$$

so that in Lorentz units of electromagnetic quantities, employing the second as the unit of time, the "electromagnetic momentum" per unit volume is $c^{-1}[\mathbf{e}\cdot\mathbf{h}] = \mathbf{g}$, while the "Poynting vector" of energy-current $\mathbf{s}$ density is $c[\mathbf{e}\cdot\mathbf{h}]$, or $c^2\mathbf{g}$, which, of course, satisfies dimensional requirements; for as $[\mathbf{e}\cdot\mathbf{h}]$ has the same dimensions as $e^2 + h^2$, i.e., energy-density, therefore $c^{-1}[\mathbf{e}\cdot\mathbf{h}]$ has dimensions of momentum-density (since c is a velocity) and $c[\mathbf{e}\cdot\mathbf{h}]$ has dimensions of flow of energy per sec. per unit area of cross-section of the energy radiation beam.

These interpretations of equations (12) have lost some of their former importance in view of the irrelevance of the concept of an absolute space filled with an ether stagnant or "practically" stagnant. If it be desirable to retain the concept of an ether, so as to give the tyro in Physics a working model to correlate the phenomena of optics and electromagnetics with those of deformable media, then we must be prepared to allow every observer to "carry his own ether about with him." For our immediate purpose the interest in equations (12) centres in the fact of their relativity; if they are true in the frame S, equations of the same form are true in S′, provided equations (6) and (7) form the basis for transformation of the measures of the physical quantities involved. We might assume this right away on the ground of the relativity of equations (1), (2), (3), (4), from which (12) can be derived; but the fact can be

most readily perceived when we appreciate that (12) can be written in the form

$$\partial E_{\lambda a}/\partial x_a = J_a F_{\lambda a} \qquad \text{(12A)}$$

or

$$\text{Lor}\ \mathbf{\mathfrak{E}} = [\mathbf{J} \cdot \mathbf{\mathfrak{F}}] \qquad \text{(12B)}$$

and can be demonstrated with little trouble.

APPENDIX TO CHAPTER VI.

Electromagnetic Phenomena in a Moving Material Medium.

We are now in a position to treat in a more complete fashion the subject dealt with in the appendix to Chapter IV. Just as we obtained in (6A) and (7A) above the three-vectorial form of equations (7), (8), and (9) of that chapter, so we can also easily write down the three-vectorial form of equations (20)-(24) of the appendix. They are :

$$\mathbf{e}' = \alpha\{\mathbf{e} + [\mathbf{u} \, . \, \mathbf{b}]\} - (\alpha - 1)(\mathbf{u} \, . \, \mathbf{e})\mathbf{u}/u^2 \quad . \quad (13)$$
$$\mathbf{d}' = \alpha\{\mathbf{d} + [\mathbf{u} \, . \, \mathbf{h}]\} - (\alpha - 1)(\mathbf{u} \, . \, \mathbf{d})\mathbf{u}/u^2 \quad . \quad (14)$$
$$\mathbf{h}' = \alpha\{\mathbf{h} - [\mathbf{u} \, . \, \mathbf{d}]\} - (\alpha - 1)(\mathbf{u} \, . \, \mathbf{h})\mathbf{u}/u^2 \quad . \quad (15)$$
$$\mathbf{b}' = \alpha\{\mathbf{b} - [\mathbf{u} \, . \, \mathbf{e}]\} - (\alpha - 1)(\mathbf{u} \, . \, \mathbf{b})\mathbf{u}/u^2 \quad . \quad (16)$$
$$\mathbf{j}' = \mathbf{j} - \alpha\rho\mathbf{u} + (\alpha - 1)(\mathbf{u} \, . \, \mathbf{j})\mathbf{u}/u^2 \quad . \quad . \quad (17)$$
$$\rho' = \alpha\{\rho - (\mathbf{u} \, . \, \mathbf{j})\}. \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Suppose now that the medium is at rest in the frame of reference S′, and that we assume the usual constitutive equations for an isotropic medium at rest, viz.,

$$\mathbf{d}' = \epsilon\mathbf{e}'$$
$$\mathbf{b}' = \mu\mathbf{h}'$$
$$\mathbf{j}' = \sigma\mathbf{e}'.$$

The equations for the medium in motion with a *uniform* velocity $\mathbf{u}$ can be easily written down, assuming that the Relativity principle is true. Some simplification is possible if we remember that $\mathbf{u}$ is at right angles to $[\mathbf{u} \, . \, \mathbf{a}]$, where $\mathbf{a}$ is any vector, and so make use of the fact that $(\mathbf{u} \, . \, [\mathbf{u} \, . \, \mathbf{a}]) = 0$. We first of all obtain

$$\mathbf{d} + [\mathbf{u} \, . \, \mathbf{h}] = \epsilon(\mathbf{e} + [\mathbf{u} \, . \, \mathbf{b}]) \quad . \quad . \quad (19)$$
$$\mathbf{b} - [\mathbf{u} \, . \, \mathbf{e}] = \mu(\mathbf{h} - [\mathbf{u} \, . \, \mathbf{d}]). \quad . \quad . \quad (20)$$

To obtain the constitutive equation for current and electric vectors we substitute in $\mathbf{j}' = \sigma\mathbf{e}'$ from (13) and (17). Then

take the scalar product of $\mathbf{u}$ and each side of this equation; this gives

$$\alpha\{(\mathbf{u} \cdot \mathbf{j}) - \rho(\mathbf{u} \cdot \mathbf{u})\} = \sigma(\mathbf{u} \cdot \mathbf{e}).$$

Introduce this value for $(\mathbf{u} \cdot \mathbf{j})$ in the left-hand side of the previous equation, and we obtain

$$\mathbf{j} - \rho\mathbf{u} = \sigma\alpha\{\mathbf{e} + [\mathbf{u} \cdot \mathbf{b}] - (\mathbf{u} \cdot \mathbf{e})\mathbf{u}\} \quad . \quad . \quad (21)$$

These constitutive equations (19), (20), (21), combined with the usual field equations

$$\begin{aligned} \partial\mathbf{d}/\partial t + \mathbf{j} &= \text{curl } \mathbf{h} \\ \text{div } \mathbf{d} &= \rho \\ \partial\mathbf{b}/\partial t &= -\text{ curl } \mathbf{e} \\ \text{div } \mathbf{b} &= 0, \end{aligned}$$

satisfy the principle of Relativity for a medium which is moving with a uniform velocity in any of the frames of reference considered.

These constitutive equations are known also not to be at variance with any experimental evidence to hand.

Thus by (19) and (20), if we neglect the square of u (and terms involving u^2 are inappreciable in the experiments) we have

$$\mathbf{d} = \epsilon\mathbf{e} + (\epsilon\mu - 1)[\mathbf{u} \cdot \mathbf{h}] \quad . \quad . \quad . \quad (22)$$

This result has been verified by Wilson, as already mentioned in Chapter V.; at least, he has verified the factor $\epsilon - 1$, for in his experiment μ was practically unity.

We also obtain

$$\mathbf{b} = \mu\mathbf{h} - (\epsilon\mu - 1)[\mathbf{u} \cdot \mathbf{e}] \quad . \quad . \quad . \quad (23)$$

This equation implies that if an electrically polarised medium be in motion, then a magnetic field proportional to the vector product of the motion and field, and also to the factor $\epsilon\mu - 1$, should be produced. Such a magnetic field was first detected by Röntgen,* and the motion of the polarised medium is therefore said to give rise to the *Röntgen current*. The verification of the factor $\epsilon\mu - 1$, or rather $\epsilon - 1$, was effected by Eichenwald.†

As regards equation (21), its left-hand side is the *conduction*

* "Berl. Sitz." (1885), p. 195; "Wied. Ann.," **35** (1888), p. 264; **40** (1890), p. 93.

† "Ann. der Phys.," **11** (1903), p. 421.

current; for $\mathbf{j}$ represents the total current and $\rho\mathbf{u}$ is the convection current. If we consider the quotient of its resolved part in any direction by the resolved part of the vector $\mathbf{e} + [\mathbf{u} \cdot \mathbf{b}]$ as the conductivity of the medium in that direction from the standpoint of an observer relative to whom the medium has a velocity $\mathbf{u}$, it appears that this conductivity has different values in different directions. Thus, in the direction of $\mathbf{u}$, the value is σ/a; in any direction at right angles to $\mathbf{u}$ it is σa.

From equation (18) we see that even if $\rho' = 0$, ρ is not zero if a conduction current is observed in the medium by those to whom it is at rest; for

$$\rho = a\{\rho' + (\mathbf{u} \cdot \mathbf{j})\}.$$

In fact, if $\rho' = 0$, then

$$\begin{aligned} \rho &= (\mathbf{u} \cdot \mathbf{j}) \\ &= (\mathbf{u} \cdot \mathbf{j} - \rho\mathbf{u}) + \rho(\mathbf{u} \cdot \mathbf{u}) \\ &= (\mathbf{u} \cdot \mathbf{i}) + \rho u^2 \end{aligned}$$

and therefore $$\rho = a^2(\mathbf{u} \cdot \mathbf{i}), \qquad (24)$$

where $\mathbf{i} = \mathbf{j} - \rho\mathbf{u}$, the conduction current in S.

The existence of such a charge was first suspected by Budde,* and a formula of a similar type plays a part in Lorentz' analysis. It is usually called the "*conduction*" or "*compensation*" charge. Of course, if $\mathbf{j}' = 0$, we have $\rho = a\rho'$, and this leads, as shown already, to invariance of the charge in a definite part of the medium; but it is clear that this invariance of charge in the *macroscopic* sense does not hold if conduction currents exist for observers in the rest-system.

So far, the limitation that the medium moves with a uniform velocity in any of the frames of reference has been imposed on the analysis, but as Cunningham has pointed out in his book (Chapter X.), it is possible to treat the problem in a more general way and remove this restriction. Before doing so, however, it will be of interest at the moment, and serve for purpose of illustration later, if we indicate briefly two of the best-known methods of dealing with the question of moving media published before the enunciation of the Relativity principle and Minkowski's application of it to this question. They are due to Hertz and Lorentz respectively.

It is, of course, well known that the "curl" equations of the field are the differential form of the integral equations which express the laws of Ampère and Faraday in their most general

* "Wied. Ann.," **10**, p. 553.

form, viz.,

$$\partial/\partial t \,.\iint(\mathbf{d}\,.\,d\mathbf{S}) + \iint(\mathbf{i}\,.\,d\mathbf{S}) = \int(\mathbf{h}\,.\,d\mathbf{s})$$
$$-\,\partial/\partial t \,.\iint(\mathbf{b}\,.\,d\mathbf{S}) = \int(\mathbf{e}\,.\,d\mathbf{s}),$$

where $\mathbf{i}$ is *conduction* current and it is understood that the surface $\mathbf{S}$ and its boundary $\mathbf{s}$ are fixed with reference to the axes. Now Hertz, regarding the ether as an absolutely fixed frame of reference, considered that when dealing with a surface $\mathbf{S}$ and its boundary $\mathbf{s}$ fixed in a body which is moving relative to the ether, one ought to replace $\partial/\partial t$ by $D/\partial t$,* thus taking account of a change in $\mathbf{d}$ due to the motion of $\mathbf{s}$ and the (spatial) non-uniformity of $\mathbf{d}$, as well as of the purely local time rate $\partial/\partial t$. Now it can be shown that

$$D/\partial t \,.\iint(\mathbf{a}\,.\,d\mathbf{S}) = \iint(\{\partial\mathbf{a}/\partial t + \mathbf{v}\operatorname{div}\mathbf{a} - \operatorname{curl}[\mathbf{v}\,.\,\mathbf{a}]\}\,.\,d\mathbf{S}),$$

if $\mathbf{v}$ be the velocity of the body at the point where a given vector has the value $\mathbf{a}$. Hence Hertz' assumption leads readily to

$$\iint(\{\partial\mathbf{d}/\partial t + \mathbf{v}\operatorname{div}\mathbf{d} - \operatorname{curl}[\mathbf{v}\,.\,\mathbf{d}] + \mathbf{i}\}\,.\,d\mathbf{S})$$
$$= \int(\mathbf{h}\,.\,d\mathbf{s}) = \iint(\operatorname{curl}\,\mathbf{h}\,.\,d\mathbf{S})$$

by Stoke's Theorem, and a similar equation for $\mathbf{b}$.

So that Hertz' four equations of the field are

$$\left.\begin{aligned} \partial\mathbf{d}/\partial t + \mathbf{i} + \rho\mathbf{v} &= \operatorname{curl}\{\mathbf{h} + [\mathbf{v}\,.\,\mathbf{d}]\\ \operatorname{div}\mathbf{d} &= \rho\\ \partial\mathbf{b}/\partial t &= -\operatorname{curl}\{\mathbf{e} - [\mathbf{v}\,.\,\mathbf{b}]\}\\ \operatorname{div}\mathbf{b} &= 0. \end{aligned}\right\} \qquad (25)$$

In framing his constitutive equations Hertz regarded the effective polarising electric intensity as the sum of $\mathbf{e}$, and an intensity $[\mathbf{v}\,.\,\mathbf{b}]$ due to the motion of the medium through the magnetic field $\mathbf{b}$, and similarly that the effective polarising magnetic intensity is the sum of $\mathbf{h}$ and an intensity $-[\mathbf{v}\,.\,\mathbf{d}]$, due to the motion of the medium through the electric field $\mathbf{d}$. Thus he writes

* If the reader is unfamiliar with the meaning of the operator $D/\partial t$, he will find it explained more fully in the succeeding chapter. Whereas $\partial/\partial t$ refers to a rate of variation at an element of surface or volume fixed relative to the axes, $D/\partial t$ refers to a rate of variation at an element moving relative to the axes, and possibly changing in size also. In fact, $D\phi(x, y, z, t)/\partial t = \partial\phi/\partial t + (\mathbf{v}\,.\,\operatorname{grad}\phi) + \phi\operatorname{div}\mathbf{v}$.

$$\left.\begin{aligned}\mathbf{d} &= \epsilon\{\mathbf{e} + [\mathbf{v} \,.\, \mathbf{b}]\} \\ \mathbf{b} &= \mu\{\mathbf{h} - [\mathbf{v} \,.\, \mathbf{d}]\}\end{aligned}\right\} \quad . \quad . \quad . \quad (26)$$

Now, quite apart from the question whether these equations satisfy the Relativity principle or not, it is certain that equations (26), which lead (to the first order in $\mathbf{v}$) to

$$\begin{aligned}\mathbf{d} &= \epsilon\mathbf{e} + \epsilon[\mathbf{v} \,.\, \mathbf{h}] \\ \mathbf{b} &= \mathbf{h} - \epsilon[\mathbf{v} \,.\, \mathbf{e}]\end{aligned}$$

(μ being practically unity), are contradicted by the experiments of Wilson and Eichenwald.

Lorentz' method was based on the theory of electrons and an averaging or "smoothing out" process, in which he takes account of the motion of "free" electrons relative to the medium as the conduction current $\mathbf{i}$ and the motion of "bound" electrons relative to the medium and of the medium relative to the ether as the convection current. Also, the "displacement" $\mathbf{d}$ is the sum of the field intensity $\mathbf{e}$ and the polarisation $\mathbf{p}$. Of course, all these symbols refer now to macroscopic averaged values and not to the microscopic values, for which Lorentz assumes as his fundamental equations those written in (1), (2), (3), (4) of the chapter. The macroscopic equations obtained finally by Lorentz are:

$$\left.\begin{aligned}\partial\mathbf{d}/\partial t + \mathbf{i} + \rho\mathbf{v} &= \operatorname{curl}\{\mathbf{h} + [\mathbf{v} \,.\, \mathbf{p}]\} \\ \operatorname{div} \mathbf{d} &= \rho \\ \partial\mathbf{b}/\partial t &= -\operatorname{curl} \mathbf{e} \\ \operatorname{div} \mathbf{b} &= 0.\end{aligned}\right\} \quad . \quad (27)$$

As his constitutive equations Lorentz takes

$$\left.\begin{aligned}\mathbf{d} + [\mathbf{v} \,.\, \mathbf{b}] &= \epsilon\{\mathbf{e} + [\mathbf{v} \,.\, \mathbf{b}]\} \\ \mathbf{b} - [\mathbf{v} \,.\, \mathbf{d}] &= \epsilon\{\mathbf{h} - [\mathbf{v} \,.\, \mathbf{d}]\}\end{aligned}\right\} \quad . \quad . \quad (28)$$

assuming that the term $[\mathbf{v} \cdot \mathbf{b}]$ will affect both displacement and intensity, and similarly for the term $-[\mathbf{v} \cdot \mathbf{d}]$.

These constitutive equations of Lorentz do not contradict the Wilson and Eichenwald results.

To investigate how far these equations of Hertz and Lorentz (or any others which have been suggested) agree with the Relativity principle (irrespective of their truth to nature), let us adopt Cunningham's method of presenting Minkowski's treatment. Taking for granted that there are five vectors, $\mathbf{d}^\times, \mathbf{e}^\times, \mathbf{b}^\times, \mathbf{h}^\times, \mathbf{j}^\times$, which are connected by the four differential equations in a given frame of reference:

$$\left.\begin{aligned} \partial \mathbf{d}^{\times}/\partial t + \mathbf{j}^{\times} &= \operatorname{curl} \mathbf{h}^{\times} \\ \operatorname{div} \mathbf{d}^{\times} &= \rho \\ \partial \mathbf{b}^{\times}/\partial t &= -\operatorname{curl} \mathbf{e}^{\times} \\ \operatorname{div} \mathbf{b}^{\times} &= 0 \end{aligned}\right\} \quad . \quad . \quad (29)\ *$$

we wish to know if similar equations are valid in any other frame of reference moving with a uniform velocity with respect to the first. (Note that this puts no limitation on the movement of the *medium* relative to any of the frames.) From the treatment in the appendix to Chapter IV., and at the beginning of this appendix, the question is answered in the affirmative; but we can drive the point home still more clearly and open up the way for a relativistic treatment of the constitutive equations by availing ourselves of the methods of Tensor Analysis. Thus let us consider the four six-vectors:

$$\begin{aligned} \mathfrak{F}_1 &= h_x^{\times}, h_y^{\times}, h_z^{\times}, -\iota d_x^{\times}, -\iota d_y^{\times}, -\iota d_z^{\times} \\ &= (\mathbf{h}^{\times}, -\iota\mathbf{d}^{\times}) \\ \mathfrak{F}_2 &= (\mathbf{b}^{\times}, -\iota\mathbf{e}^{\times}) \\ \mathfrak{R}_1 &= -(\mathbf{e}^{\times}, \iota\mathbf{b}^{\times}) \\ \mathfrak{R}_2 &= -(\mathbf{d}^{\times}, \iota\mathbf{h}^{\times}) \end{aligned}$$

and the four-vector

$$\mathbf{J} = (\mathbf{j}, \iota\rho).$$

Equations (29) can be written in the form

$$\left.\begin{aligned} \operatorname{Lor} \mathfrak{F}_1 &= \mathbf{J} \\ \operatorname{Lor} \mathfrak{R}_1 &= 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (30)$$

which justifies their claim to agreement with Relativity, provided $\mathfrak{F}_1$, etc., are six-vectors and $\mathbf{J}$ a four-vector, i.e., provided there are the usual linear relations between fields, current and density as measured in two frames of reference in uniform relative motion.

In order to construct relativistic constitutive equations, let us consider the following four-vector products of the four-vector:

$$\mathbf{V} = \beta(\mathbf{v}, \iota)$$

and the field six-vectors, $\mathfrak{F}_1$, etc.

* We write the small cross × after each symbol so as to avoid any implication at the outset that it represents the same physical vector as the corresponding symbol does in any other treatment.

$$\begin{aligned}
\mathbf{D} &= [\mathbf{V} \cdot \mathfrak{F}_1] = \beta\{\mathbf{d}^\times + [\mathbf{v} \cdot \mathbf{h}^\times],\ \iota(\mathbf{v} \cdot \mathbf{d}^\times)\} \\
\mathbf{E} &= [\mathbf{V} \cdot \mathfrak{F}_2] = \beta\{\mathbf{e}^\times + [\mathbf{v} \cdot \mathbf{b}^\times],\ \iota(\mathbf{v} \cdot \mathbf{e}^\times)\} \\
\mathbf{B} &= [\mathbf{V} \cdot \mathfrak{R}_1] = \beta\{\mathbf{b}^\times - [\mathbf{v} \cdot \mathbf{e}^\times],\ \iota(\mathbf{v} \cdot \mathbf{b}^\times)\} \\
\mathbf{H} &= [\mathbf{V} \cdot \mathfrak{R}_2] = \beta\{\mathbf{h}^\times - [\mathbf{v} \cdot \mathbf{d}^\times],\ \iota(\mathbf{v} \cdot \mathbf{h}^\times)\}.^*
\end{aligned}$$

Now any linear relation between **D** and **E**, or between **B** and **H** would, of course, be a constitutive equation which would satisfy the principle of Relativity. As to its truth to nature, that would be a matter for experiment; but as a first attempt it is obviously natural to write down one which degenerates to the usual type for a medium at rest in a frame. Such a pair are clearly

and
$$\left.\begin{aligned} \mathbf{D} &= \epsilon \mathbf{E} \\ \mathbf{B} &= \mu \mathbf{H} \end{aligned}\right\} \qquad (31)$$

In terms of the field three-vectors these can be seen to be equivalent to

$$\left.\begin{aligned} \mathbf{d}^\times + [\mathbf{v} \cdot \mathbf{h}^\times] &= \epsilon\{\mathbf{e}^\times + [\mathbf{v} \cdot \mathbf{b}^\times]\} \\ \mathbf{b}^\times - [\mathbf{v} \cdot \mathbf{e}^\times] &= \mu\{\mathbf{h}^\times - [\mathbf{v} \cdot \mathbf{d}^\times]\} \end{aligned}\right\} \qquad (32)$$

for these imply that

and
$$\begin{aligned} (\mathbf{v} \cdot \mathbf{d}^\times) &= \epsilon(\mathbf{v} \cdot \mathbf{e}^\times) \\ (\mathbf{v} \cdot \mathbf{b}^\times) &= \mu(\mathbf{v} \cdot \mathbf{h}^\times), \end{aligned}$$

and so comply with the complete equalities in (31).

For the third constitutive relation it would seem natural to write

$$\mathbf{J} = \sigma \mathbf{E},$$

but since $(\mathbf{V} \cdot \mathbf{E}) = 0$, this would necessitate $(\mathbf{V} \cdot \mathbf{J}) = 0$, which is not generally true. We can get round this difficulty by resolving **J** (four-dimensionally) into two components parallel and perpendicular to **V**, thus:

$$\mathbf{J} = \mathbf{I} + \lambda \mathbf{V},$$

where $(\mathbf{V} \cdot \mathbf{I}) = 0$, and λ is a multiplier to be determined.

Since
$$\begin{aligned} (\mathbf{V} \cdot \mathbf{I}) &= 0 \text{ by hypothesis,} \\ (\mathbf{V} \cdot \mathbf{J}) &= \lambda(\mathbf{V} \cdot \mathbf{V}) \\ &= \lambda(\beta^2 v^2 - v^2) \\ &= -\lambda. \end{aligned}$$

* We write these as **D**, **E**, etc., in accordance with our convention for symbols representing four-vectors. They must not, of course, be confused with the field three-vectors.

Also $(\mathbf{V} \cdot \mathbf{J})$ is an invariant, so we can calculate its value by referring to a frame in which the point in question is at rest, in which case $\mathbf{V} = (0, 0, 0, \iota)$ and $\mathbf{J} = (j_{ox}, j_{oy}, j_{oz}, \iota\rho_o)$, ρ_o being the "proper" or "rest" density of charge.

Hence

$$(\mathbf{V} \cdot \mathbf{J}) = -\rho_o,$$

and so

$$\lambda = \rho_o.$$

Hence

$$\mathbf{J} = \mathbf{I} + \rho_o\mathbf{V}.$$

We can now assume for the third constitutive relation

$$\mathbf{I} = \mathbf{J} - \rho_o\mathbf{V} = \sigma\mathbf{E}.$$

This leads to the three-vectorial equation

$$\mathbf{j}^\times - \beta\rho_o\mathbf{v} = \beta\sigma\{\mathbf{e}^\times + [\mathbf{v} \cdot \mathbf{b}^\times]\},$$

and also to

$$\iota\rho = \iota\beta\sigma(\mathbf{v} \cdot \mathbf{e}^\times) + \iota\beta\rho_o,$$

so that

$$\beta\rho_o\mathbf{v} = \rho\mathbf{v} - \beta\sigma(\mathbf{v} \cdot \mathbf{e}^\times)\,\mathbf{v}.$$

Hence the assumed constitutive equation becomes finally

$$\mathbf{j}^\times - \rho\mathbf{v} = \beta\sigma\{\mathbf{e}^\times + [\mathbf{v} \cdot \mathbf{b}^+] - (\mathbf{v} \cdot \mathbf{e}^+)\mathbf{v}\} \quad . \quad (33)$$

We are now in a position to test the theories of Hertz and Lorentz, from the point of view of their ability to pass the Relativity test. In the case of Hertz' equations, we write

$$\begin{aligned} \mathbf{d} &= \mathbf{d}^\times \\ \mathbf{b} &= \mathbf{b}^\times \\ \mathbf{h} + [\mathbf{v} \cdot \mathbf{d}] &= \mathbf{h}^\times \\ \mathbf{e} - [\mathbf{v} \cdot \mathbf{b}] &= \mathbf{e}^\times \\ \mathbf{j} &= \mathbf{i} + \rho\mathbf{v} = \mathbf{j}^\times, \end{aligned}$$

and thus the field-equations of Hertz are relativistic. As regards the constitutive equations, however, these ought to be

$$\mathbf{d} + [\mathbf{v} \cdot \mathbf{h}] + [\mathbf{v} \cdot [\mathbf{v} \cdot \mathbf{d}]] = \epsilon\mathbf{e},$$

or to the first order in $\mathbf{v}$,

$$\mathbf{d} + [\mathbf{v} \cdot \mathbf{h}] = \epsilon\mathbf{e}.$$

Such a constitutive equation would comply with Relativity, but it does not agree with experiment (any more than Hertz'

own equation, which is not even relativistic), at least if $\mathbf{e}$ is interpreted in the usual way, viz., that the difference of electric potential between two points A and B is measured by $\int_{AB}(\mathbf{e} \cdot d\mathbf{s})$.

Similarly a second relativistic constitutive equation suitable for Hertz' equations would be

$$\mathbf{b} - [\mathbf{v} \cdot \mathbf{e}] + [\mathbf{v} \cdot [\mathbf{v} \cdot \mathbf{b}]] = \mu\mathbf{h},$$

or

$$\mathbf{b} - (\mathbf{v} \cdot \mathbf{e}) = \mu\mathbf{h}$$

(to the first order), to which similar remarks apply.

To test Lorentz' equations we put

$$\begin{aligned} \mathbf{d} &= \mathbf{d}^{\times} \\ \mathbf{e} &= \mathbf{e}^{\times} \\ \mathbf{b} &= \mathbf{b}^{\times} \end{aligned}$$

and

$$\begin{aligned} \mathbf{h} + [\mathbf{v} \cdot \mathbf{p}] &= \mathbf{h}^{\times} \text{ (where } \mathbf{p} = \mathbf{d} - \mathbf{e}) \\ \mathbf{j} = \mathbf{i} + \rho\mathbf{v} &= \mathbf{j}^{\times}. \end{aligned}$$

As before the field equations are clearly relativistic. The relativistic constitutive equations derived from (31) are

$$\mathbf{d} + [\mathbf{v} \cdot \mathbf{h}] + [\mathbf{v} \cdot [\mathbf{v} \cdot \mathbf{p}]] = \epsilon\{\mathbf{e} + [\mathbf{v} \cdot \mathbf{b}]\}$$

and

$$\begin{aligned} \mathbf{b} - [\mathbf{v} \cdot \mathbf{e}] &= \mu\{\mathbf{h} + [\mathbf{v} \cdot \mathbf{p}] - [\mathbf{v} \cdot \mathbf{d}]\} \\ &= \mu\{\mathbf{h} - [\mathbf{v} \cdot \mathbf{e}]\}, \end{aligned}$$

or to the first order

$$\mathbf{d} = \epsilon\{\mathbf{e} + [\mathbf{v} \cdot \mathbf{b}]\} - [\mathbf{v} \cdot \mathbf{h}]$$

and

$$\mathbf{b} = \mu\mathbf{h} - (\mu - 1)[\mathbf{v} \cdot \mathbf{e}],$$

with which Lorentz' equations agree to the order required, and on the assumption that μ is practically unity.

Of course, still more general linear relations could have been assumed in the first instance instead of (31), and would certainly have to be assumed in the case of æolotropic media. The relativistic course of procedure would be to write

$$\mathbf{D} = [\epsilon \cdot \mathbf{E}],$$

where ϵ would be now regarded as a tensor-operator of the second order. This means that we write

$$\begin{aligned} D_1 &= \epsilon_{11}E_1 + \epsilon_{12}E_2 + \epsilon_{13}E_3 + \epsilon_{14}E_4 \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ D_4 &= \epsilon_{41}E_1 + \epsilon_{42}E_2 + \epsilon_{43}E_3 + \epsilon_{41}E_4 \end{aligned}$$

and assume that the $\epsilon_{\lambda\mu}$ transform by the relations

$$\epsilon_{\lambda\mu}{}' = a_{\lambda\alpha}\, a_{\mu\beta}\, \epsilon_{\lambda\mu},$$

which would ensure that in a new frame

$$D_1{}' = \epsilon_{11}{}'E_1{}' + \epsilon_{12}{}'E_2{}' + \epsilon_{13}{}'E_3{}' + \epsilon_{14}{}'E_4{}',$$

etc.

The subject matter of this appendix is an excellent illustration of the method of Relativity, not as a means of actually giving a correct equation or law, but as a test between different alternatives.

CHAPTER VII.

RELATIVITY AND THE CONSERVATION LAWS.

THE relativity of equation (12) of the last chapter, and the attempt to interpret it in terms of purely mechanical concepts such as " momentum," " force," " energy," naturally leads us to enquire as to the possible relativity of similar relations (i.e., conservation laws) in strictly mechanical phenomena. We have worked out the conditions which " mass " and " force " must be subject to, in order to satisfy the postulate of Relativity in particle dynamics, where bodies are treated as indivisible things, whose rotation or internal condition is of no immediate importance. It is an obvious extension of our work to consider the bearing of Relativity on the dynamics of continuous media, where internal state is considered. (In Relativity mechanics we cannot speak of " Dynamics of *Rigid* Bodies," as the concept of rigidity or undeformability must disappear in view of the alteration of size and shape which accompanies an alteration of velocity relative to any chosen frame.) It is here that equations (12) can render us some assistance. At the time of their discovery it was quite natural to attempt to interpret them mechanically; now we may reverse the process and use them to suggest to us the form in which the dynamical equations for continuous media should be exhibited so that their relativity may be most directly demonstrated.

We know that the laws of conservation of momentum and energy, when expressed analytically, give a comprehensive basis for general dynamics. We must be prepared in Relativity dynamics for the identification of mass and energy already suggested by the dynamics of a particle, and also for variation of the mass (and energy) of portions of matter which we have been in the habit of segregating as discrete bodies, bounded by conceptual surfaces, which are the loci of points representing presumably identifiable particles. The results just obtained in electromagnetic theory are bound to react on our ideas and invite us to test whether the principle of Relativity does in any

way call into question conservation of energy and momentum, in the sense that if demonstrated as true in one frame of reference it might not be true in another. In discussing the laws of motion for continuous media, we are accustomed to make use of the concept of the stress at a point in the medium, analysed into nine components:

$$\begin{matrix} t_{xx} & t_{xy} & t_{xz} \\ t_{yx} & t_{yy} & t_{yz} \\ t_{zx} & t_{zy} & t_{zz} \end{matrix}$$

where t_{xx} is the tension per unit area exerted across a plane perpendicular to OX in the direction OX, t_{xy} is the shearing stress per unit area exerted across a plane perpendicular to OY in the direction OX, t_{yx}, across a plane perpendicular to OX in the direction OY, etc. This is, in fact, the three-dimensional "stress-tensor" well known to readers of text-books of Elasticity. When the medium is in motion in a frame S, each portion of the medium possesses momentum and also energy. So we can speak of a "density of momentum" $\mathbf{g}$ at a point in the sense that if $\delta\tau$ is an element of volume, containing the point, $\mathbf{g}\delta\tau$ is its momentum; and of a "density of energy" μ, where $\mu\delta\tau$ is the energy of the volume element. ($\mu\delta\tau$ may be kinetic energy, potential energy due to body forces, heat energy, or electromagnetic energy.)

On such an element $\delta\tau$ there may also be a "body-force" exerted due to external causes and not arising from the stresses. This we will denote by $\mathbf{k}\delta\tau$, so that $\mathbf{k}$ is the force per unit volume of the medium or "force-density."

It is shown in text-books of Elasticity and Hydrodynamics that the force on an element arising from the internal stresses is given by

$$\delta\tau(\partial t_{xx}/\partial x + \partial t_{xy}/\partial y + \partial t_{xz}/\partial z)$$

in the direction OX and two similar expressions, so that the total force is given by

$$\delta\tau(k_x + \Sigma\partial t_{xx}/\partial x)$$

in the direction OX and two similar expressions.

In Chapters III. and VI. it appeared that the verbal form of the second law of motion could still be retained in Relativity dynamics of a particle, provided the notions of mass and momentum were somewhat generalised, and the expressions for the force in different frames related to one another in a particular

way. Our natural desire is to retain the law also in the dynamics of a continuous medium, and investigate the conditions to which our concepts of momentum, energy, and force must be subject in order to satisfy the Relativity principle. We have just written down the expression for the total force on an element of volume, and we must obtain the rate of change of momentum of the element. It will not do merely to write down $\delta\tau\partial\mathbf{g}/\partial t$. That served our purpose in equation (12); but, as was pointed out, the mechanical interpretation of (12) involved the assumption that the ether was practically immobile. But we are now dealing with a moving medium, and $\delta\tau\partial\mathbf{g}/\partial t$ would refer to the rate of change of momentum within an unchanging element of volume fixed in S, since $\partial/\partial t$ implies differentiation with regard to t, the co-ordinates x, y, z remaining constant. Actually $\delta\tau$ is moving in S and generally altering in size. What we want is $d(\mathbf{g}\delta\tau)/dt$, taking account of change of position and size on the part of $\delta\tau$. One portion of this is $\delta\tau\partial\mathbf{g}/\partial t$. A second part, due to flow of momentum into $\delta\tau$ across its surface, is given by

$$\delta\tau(v_x\partial g_x/\partial x + v_y\partial g_x/\partial y + v_z\partial g_x/\partial z)$$

and two similar expressions, or

$$\delta\tau(\mathbf{v}\ .\ \mathrm{grad}\ g_x),\ \delta\tau(\mathbf{v}\ .\ \mathrm{grad}\ g_y),\ \delta\tau(\mathbf{v}\ .\ \mathrm{grad}\ g_z).$$

A third part is due to the change of size in $\delta\tau$, and is given by

$$\delta\tau g_x(\partial v_x/\partial x + \partial v_y/\partial y + \partial v_z/\partial z)$$

and two similar expressions; or

$$\delta\tau\mathbf{g}\ \mathrm{div}\ \mathbf{v}.$$

Adding together the three sets of expressions and equating them to the total force on the element $\delta\tau$, we obtain

$$k_x + \Sigma(\partial t_{xx}/\partial x) = \mathrm{D}g_x/\partial t,$$

and two similar equations where the operator $\mathrm{D}/\partial t$ is defined by the equation

$$\begin{aligned}\mathrm{D}\phi(x, y, z, t)/\partial t &= \partial\phi/\partial t + (\mathbf{v}\ .\ \mathrm{grad}\ \phi) + \phi\ \mathrm{div}\ \mathbf{v}\\ &= \partial\phi/\partial t + \mathrm{div}\ (\phi\mathbf{v}).\end{aligned}$$

In the same way the conservation of energy is expressed by equating the sum of the activities of the body forces and the stresses on the element $\delta\tau$ to the rate at which the energy in

the moving element is changing. This equation is

$$(\mathbf{v} \cdot \mathbf{k}) + \Sigma\{\partial(v_x t_{xx} + v_y t_{yx} + v_z t_{zx})/\partial x\} = \mathrm{D}\mu/\partial t.$$

It will be a little more convenient to introduce normal and shearing components of pressure rather than tension and write $p_{xx} = - t_{xx}$, $p_{xy} = - t_{xy}$, etc. It should also be observed that $v_x p_{xx} + v_y p_{yx} + v_z p_{zx}$, and two similar expressions, are the components of a three-vector, viz., the vector-product of the vector $\mathbf{v}$ and the stress tensor $\mathbf{p}$, which we can represent by the symbol $[\mathbf{v} \cdot \mathbf{p}]$. It is not very troublesome to establish that the equations just obtained can be written in the form

$$\mathrm{Lor}\ \mathfrak{T} = \mathbf{K},$$

where $\mathfrak{T}$ represents the array of sixteen quantities:

$$\left.\begin{array}{llll} T_{11}=p_{xx}+g_x v_x, & T_{12}=p_{xy}+g_x v_y, & T_{13}=p_{xz}+g_x v_z, & T_{14}=\iota g_x \\ T_{21}=p_{yx}+g_y v_x, & \cdot \quad \cdot & \cdot \quad \cdot & T_{24}=\iota g_y \\ T_{31}=p_{zx}+g_z v_x, & \cdot \quad \cdot & \cdot \quad \cdot & T_{34}=\iota g_z \\ T_{41}=\iota(\mu v_x+[\mathbf{v} \cdot \mathbf{p}]_x), & & \cdot \quad \cdot & T_{44}=-\mu \\ \quad =\iota s_x & & & \end{array}\right\} \quad (1)$$

and

$$K_1 = k_x,\ K_2 = k_y,\ K_3 = k_z,\ K_4 = \iota(\mathbf{v} \cdot \mathbf{k}), \quad . \quad (2)$$

while, as before, x_1, x_2, x_3, x_4 replace x, y, z, ιt.

We have thus arrived at the conclusion that if the general dynamics of bodies is to be included within the scope of the Relativity principle, the quantities $T_{\lambda\mu}$ are to transform as a tensor of the second order, and K_λ are to transform as a four-vector. Before proceeding to develop this conclusion still further, let us see if there is any partial support for it in our earlier dynamics.

Suppose we consider a body in an unstrained state, so that for it t_{xx}, etc., are zero. Such a body could be regarded as a collection of particles with no cohesive forces between them, which would travel over exactly similar paths under the influence of exactly equal external forces on each particle, no internal stresses being required to prevent the body disintegrating. It could be regarded simply as a large particle. Let μ_0 be the mass of unit volume of it when at rest in a frame of reference. If referred to a frame in which the body has a velocity $\mathbf{v}$, the previous unit volume becomes $1/\beta$, and the mass of this reduced volume is now $\mu_0\beta$, so that its mass per unit volume is now $\mu_0\beta^2$. Hence for such a body:

$$\mathbf{g} = \beta^2\mu_0\mathbf{v}$$
$$\epsilon = \beta^2\mu_0$$

and the quantities $T_{\lambda\mu}$ become

$$\begin{array}{llll}
\beta^2\mu_0 v_x^2 & \beta^2\mu_0 v_x v_y & \beta^2\mu_0 v_x v_z & \iota\beta^2\mu_0 v_x \\
\beta^2\mu_0 v_y v_x & \beta^2\mu_0 v_y^2 & \beta^2\mu_0 v_y v_z & \iota\beta^2\mu_0 v_y \\
\beta^2\mu_0 v_z v_x & \beta^2\mu_0 v_z v_y & \beta^2\mu_0 v_z^2 & \iota\beta^2\mu_0 v_z \\
\iota\beta^2\mu_0 v_x & \iota\beta^2\mu_0 v_y & \iota\beta^2\mu_0 v_z & -\beta^2\mu_0
\end{array}$$

Now we have seen that βv_x, βv_y, βv_z, $\iota\beta$ constitute a four-vector, and so, by reference to Chapter VI., we find that the sixteen quantities just written down form a tensor of the second order, being the tensor product of two vectors, viz., βv_x, etc., and $\mu_0\beta v_x$, etc.; for μ_0 is invariant. Further, if $\mathbf{f}$ is the force acting on the body and τ its "proper" volume, so that τ/β is its volume when its velocity is $\mathbf{v}$, then

$$\mathbf{k} = \beta\mathbf{f}/\tau.$$

From the dynamics of a particle we know that βf_x, βf_y, βf_z, $\iota\beta(\mathbf{v} \cdot \mathbf{f})$ are components of a four-vector. So also are k_x, k_y, k_z, $\iota(\mathbf{v} \cdot \mathbf{k})$, since τ is a given constant.

Thus in the case of an unstrained body we have some justification for assuming that $T_{\lambda\mu}$ and K_λ are a tensor and four-vector respectively. Indeed, we might find further support for this view, even in the case of a strained body by an appeal to the kinetic-molecular theory of matter. For we can regard p_{xx} as a flow of x-momentum per unit time across unit area normal to OX, p_{xy} as a flow of x-momentum per unit time across unit area normal to OY, etc., due to the chaotic thermal motion of the molecules with reference to axes fixed in the body. Thus $g_x v_x + p_{xx}$ would be a total flow of momentum reckoned by giving to each molecule a velocity in the frame S compounded of $\mathbf{v}$ and its thermal velocity relative to axes fixed in the body. It would be the sum of a number of terms such as $\mu_r\beta_r(v_r)_x^2$ reckoned for the individual molecules. Similarly, $g_x v_y + p_{xy}$ would be $\Sigma\mu_r\beta_r(v_r)_x(v_r)_y$, and so on. Regarding each term of these sums as coming under the conclusions arrived at for particle dynamics, we would conclude that the sums are components of a tensor; for if one adds corresponding components of a number of tensors, the result is a tensor.

Turning from such special considerations let us apply the assumption that equations of conservation satisfy the Relativity test to the case where the motion of the medium is *adiabatic.* This means that with reference to an observer who is at rest

relative to a point in the medium, the momentum-density and the energy-current density at that point are zero, so that, for instance, there is no transfer of heat relative to him at that point.

Let S′ be a space frame in which the point in question is at rest, and at that point

$$\left.\begin{array}{l} v_x' = v_y' = v_z' = 0 \\ g_x' = g_y' = g_z' = 0. \end{array}\right\} \quad . \quad . \quad . \quad (3)$$

Furthermore, by the ordinary mechanics of material media at rest we have at the point

$$t_{yz}' = t_{zy}' \,; \; t_{zx}' = t_{xz}' \,; \; t_{xy}' = t_{yx}' \quad . \quad . \quad (4)$$

Hence it appears that

$$\left.\begin{array}{l} T_{23}' = T_{32}' \,; \quad T_{31}' = T_{13}' \,; \quad T_{12}' = T_{21}' \\ T_{14}' = T_{24}' = T_{34}' = T_{41}' = T_{42}' = T_{43}' = 0. \end{array}\right\} \quad (5)$$

Hence the tensor has symmetrical components when expressed in accented co-ordinates. But symmetry is a property independent of axes. Therefore the tensor $T_{\lambda\mu}$ is symmetrical in any co-ordinates. From this it follows that

$$g_x = \mu v_x + [\mathbf{v} \cdot \mathbf{p}]_x$$

and two similar equations; or, in vectorial rotation,

$$\mathbf{g} = \mu\mathbf{v} + [\mathbf{v} \cdot \mathbf{p}] \quad . \quad . \quad . \quad (6)$$

Momentum and Velocity are not in General Co-directional.

The x-component of $[\mathbf{v} \cdot \mathbf{p}]$ represents the transference of energy *of strain* across unit area perpendicular to OX. Hence the right-hand side of (6) represents a flux of energy per unit area, and we see that it is accompanied by a density of momentum numerically equal to this flux in Relativity units. (To obtain the momentum density in C.G.S. units, we would as usual have to divide the right-hand side by c^2.) This result is a wider aspect of the identification of mass and energy already obtained in our particle dynamics. In the special case of an unstrained medium,

$$\mathbf{g} = \mu\mathbf{v}.$$

But, in general, as we may demonstrate from (6), $\mathbf{g}$ and $\mathbf{v}$

have not the same direction. This can be shown most readily by using the simple Lorentz transformation, a small portion of the body being at rest in S′ but in motion relative to S, with a velocity **u** parallel to OX. In this case the scheme of $a_{\lambda\mu}$ coefficients becomes

$$\begin{array}{cccc} a & 0 & 0 & \iota a u \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\iota a u & 0 & 0 & a \end{array}$$

where $a = (1 - u^2)^{-\frac{1}{2}}$; and by (1), (3), and (5),

$$T_{11}' = p_{xx}',\ T_{12}' = p_{xy}',\ T_{13}' = p_{xz}',\ T_{14}' = 0$$

and two similar sets; and

$$T_{41}' = 0,\ T_{42}' = 0,\ T_{43}' = 0,\ T_{44}' = -\mu'.$$

Therefore

$$\begin{aligned} T_{14} &= a_{\alpha 1} a_{\beta 4} T_{\alpha\beta}' \\ &= a_{11} a_{14} T_{11}' + a_{41} a_{44} T_{41}' \\ &= \iota a^2 u(\mu' + p_{xx}'), \\ T_{24} &= a_{\alpha 2} a_{\beta 4} T_{\alpha\beta}' \\ &= a_{22} a_{14} T_{21}' \\ &= \iota a u p_{yx}', \end{aligned}$$

and $$T_{34} = \iota a u p_{zx}',$$

i.e., $$\left.\begin{aligned} g_x &= a^2 u(\mu' + p_{xx}') \\ g_y &= a u p_{yx}' \\ g_z &= a u p_{zx}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (7)$$

Hence if the medium be strained the momentum density **g** is not in general directed along OX, i.e., along the direction of motion in S of the point in question. This will only occur if there be no stress, or if the stress components p_{yx}' and p_{zx}' vanish, in which case the state of stress for the body at rest is one of uniform pressure in all directions or simple hydrostatic pressure. This difference of direction between velocity and momentum is important, as it offers an explanation in detail of the negative result of an experiment carried out by Trouton and Noble in 1903, which, had it proved successful, would have exhibited positive evidence of terrestrial motion through the ether. We shall deal with this experiment presently, but before doing so it will be of interest to deduce one more important result from the postulated relativity of our equations. If in the frame S′ the medium at rest be subject to a uniform hydrostatic pressure p', then

$$p_{xx}' = p_{yy}' = p_{zz}' = p'$$

and the remaining components are zero. So every component of $T'_{\lambda\mu}$ is zero except four, viz.,

$$T_{11}' = T_{22}' = T_{33}' = p', \text{ and } T_{44}' = -\mu'.$$

Hence the equations of transformation give

$$\begin{aligned} T_{11} &= a_{\alpha 1}a_{\beta 1}T_{\alpha\beta}' \\ &= a_{11}a_{11}T_{11}' + a_{14}a_{14}T_{44}' \\ &= \alpha^2 p' + \alpha^2 u^2 \mu' \end{aligned}$$

i.e., $$g_x u + p_{xx} = \alpha^2 p' + \alpha^2 u^2 \mu'.$$

But by (7),

$$g_x = \alpha^2 u(\mu' + p'),$$

so that $$p_{xx} = p'.$$

Also $$\begin{aligned} T_{22} &= a_{\alpha 2}a_{\beta 2}T_{\alpha\beta}' \\ &= T_{22}' \\ &= p' \end{aligned}$$

i.e., by (1) $$p_{yy} = p'$$

similarly $$p_{zz} = p'.$$

Proceeding in this way we can likewise prove that p_{yz} etc., are zero.

Hence if a medium, which is subject to a uniform hydrostatic pressure when referred to axes with regard to which it is at rest, be in uniform motion with regard to other axes, not only is it still subject to uniform pressure, but the *measure of the pressure is invariant*, or $p = p'$.

Moreover, by (7),

$$g_x = \alpha^2(\mu' + p')u$$

and $$g_y = g_z = 0,$$

so that, as already stated, momentum and velocity have the same direction in this case. Considering a volume τ of the body in S, its momentum parallel to OX is $g_x\tau$, or $\alpha^2(\mu' + p')\tau u$. But τ', the volume of this part of the body in S′, is given by $\tau' = \alpha\tau$ Hence the momentum is $\alpha(\mu' + p')\tau' u$, or $\alpha(U' + p'\tau')u$, where U' is the internal energy of this part of the body in its rest frame. Hence for a mechanical interpretation of the momentum we are to regard the mass of the proper volume τ' of the body as

$$\alpha(U' + p'\tau')$$

when it has a velocity u. That is, we are to consider its proper mass as

$$U' + p'\tau'.$$

This is the well-known *Heat Function* of Gibbs, and the conclusion reached is in agreement with familiar notions concerning pressure, which regard it as an energy of strain per unit volume to be added to the usual internal energy, thermal, electric, etc. This result is an extension of the earlier result obtained for particle dynamics.

Trouton and Noble's Experiment.

Returning now to the experiment of Trouton and Noble* their apparatus consisted of a condenser suspended with its plates vertical by a very fine phosphor bronze strip. The charges on the plates were passed into them by means of this wire and by a wire hung from beneath and dipping into a liquid terminal. The argument (suggested by Fitzgerald) was that if this condenser were drifting with the earth's motion through the ether, with the normal to its plates inclined to the direction of drift, then there should be a mechanical effect tending to turn the normal into this direction, and a rotation should be observed when the plates were charged and discharged. This experiment differed from the other experiments on ether-drift, in that it sought for a mechanical effect as distinct from an optical or electrical one, but like them proved to be negative in its result. Had it been otherwise, it would have provided an experimental contradiction of the postulates of Relativity enunciated two years later by Einstein. The mathematical basis for the expectation ran as follows. If there be a horizontal electric intensity between the plates X, the drift of this field through the ether with a resolved velocity u at an angle θ to the normal would give rise to a magnetic field in a vertical direction equal to $Xu \sin \theta$. The energy of this magnetic field is $\frac{1}{2} X^2u^2 \sin^2 \theta$ per unit volume, and so the whole magnetic energy is $Wu^2 \sin^2 \theta$, where W is the total electrostatic energy. (This result must be slightly modified if we take account of a Lorentz-Fitzgerald contraction owing to motion through the ether, but the final result is not practically altered.) Since this magnetic energy (for constant charges on the condenser) is minimum for $\theta = 0$, there should be a couple tending to turn the condenser to this orientation equal in magnitude to

$$d(Wu \sin^2 \theta)/d\theta,$$

i.e., to

$$Wu^2 \sin 2\theta.$$

* "P.R.S.," **72**, p. 132; "Phil. Trans.," **202**, p. 163.

Now, as a matter of fact, no rotation results form the existence of this couple, and the explanation of its absence bears some analogy to the fact that in Relativity mechanics a force may not produce acceleration in a body provided its mass alters owing to loss or gain of energy by radiation, etc. We consider the condenser as at rest in a frame of reference S′ which is moving with a velocity u parallel to OX relative to the frame S (Fig. 5); the normal to the condenser makes an angle θ' with OX′. (The angle it makes with OX in S, viz. θ, differs from θ' by a quantity of the second order, but this introduces no practical error in the result. Then in S′

$$E_x' = X\cos\theta',\ E_y' = X\sin\theta',\ E_z' = 0.$$

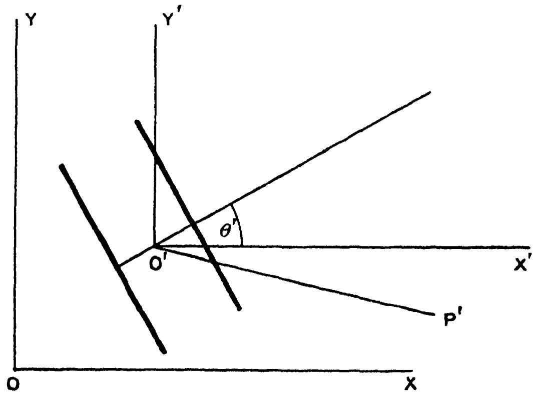

FIG. 5.

We regard the dielectric as being in a state of strain, and the stress components as given by Maxwell's formulæ:

$$\begin{aligned} t_{xx}' &= \tfrac{1}{2}(E_x'^2 - E_y'^2 - E_z'^2) = \tfrac{1}{2}X^2\cos 2\theta' \\ t_{xy}' &= E_x'E_y' = \tfrac{1}{2}X^2\sin 2\theta' \\ t_{xz}' &= E_x'E_z' = 0. \end{aligned}$$

Also, μ', the energy density, $= \frac{1}{2}X^2$, and the components of $\mathbf{g}'$, momentum density, are zero.

Referring the condenser to S, we have by (7),

$$\begin{aligned} g_x &= a^2u\mu'(1 - \cos 2\theta') \\ &= 2a^2u\mu'\sin^2\theta' \\ g_y &= -\,au\mu'\sin 2\theta' \\ g_z &= 0. \end{aligned}$$

To obtain the total momentum of the condenser in S we must remember that an element of volume $\delta x \delta y \delta z$ in S is equal to $1/a$ of the corresponding element of volume in S′.

Hence

$$\begin{aligned}\iiint g_x dxdydz &= 2au \sin^2 \theta' \iiint \mu' dx'dy'dz' \\ &= 2aWu \sin^2 \theta', \\ \iiint g_y dxdydz &= - Wu \sin 2\theta', \\ \iiint g_z dxdydz &= 0.\end{aligned}$$

If we calculate the moment of momentum about the axis OZ, it is equal to

$$\begin{aligned}&\iiint (yg_x - xg_y)dxdydz \\ &= 2\bar{y}aWu^2 \sin^2 \theta' + \bar{x}Wu \sin 2\theta',\end{aligned}$$

where $\bar{x}$, $\bar{y}$ are the co-ordinates of O′ with respect to OX, OY. Since $\bar{x}$ is changing at the rate u per unit of time, but $\bar{y}$ is constant, it appears that the rate of change of angular momentum round OZ is equal to

$$W u^2 \sin 2\theta',$$

which is just the value calculated above for the couple acting on the condenser.

It will be seen, quite apart from this detailed analysis, that the existence of a couple without accelerated rotation depends on the fact that the momentum has a direction different from that of the velocity, whereby a change in the angular momentum is produced not by a change in the momentum, but by a change in the arm of the moment of this momentum caused by the motion. Thus, in the figure, the line along which the resultant momentum is directed is O′P′, and the distance of O from this line is increasing (which would not be the case if it were O′X′), and so there is a right-handed alteration in the angular momentum about OZ, corresponding to the right-handed couple tending to diminish the magnetic energy as calculated earlier.

Relativity and Thermodynamics.

It will be of interest before leaving these considerations concerning continuous media, to deal with one or two important conclusions in Thermodynamics.

Suppose we have a thermodynamical system at rest in S′ with a uniform pressure p', a volume V′, and a uniform absolute temperature T′. We have just seen that in the frame S the

system has a uniform pressure p equal to p', and a volume V equal to V'/α. It is natural to ask what will be the effect of its motion on the measure of its temperature. If the system be a perfect gas, we can obtain an answer very simply and directly; for from the law of Boyle and Charles

$$p'V' = RT',$$

where R is a constant which (if we adopt the kinetic-molecular hypothesis of matter) is equal to Nk, where N is the number of molecules in the body of gas considered, and k is the so-called "gas constant per molecule," i.e., the quotient of the average energy of a molecule in a gas enclosed in a vessel (at rest in the frame of reference) divided by the absolute temperature (measured in this frame). Now k is a universal constant, and N, which is a number of discrete particles, is not altered by change of axes; hence R is invariant for the change of axes. But so also is p. Hence

$$pV' = RT'.$$

But if the Boyle-Charles law is to hold in S,

$$pV = RT.$$

Therefore, since $V = V'/\alpha$, it follows that

$$T = T'/\alpha,$$

or temperature is transformed like volume. In fact, αT is invariant, involving a decrease in the measure of the temperature as the velocity of the system relative to the observer is increased.

This method of drawing the conclusion is, however, unsatisfactory. It only applies to a perfect gas, and even then, one has to introduce considerations involving the kinetic-molecular hypothesis, a procedure quite foreign to reasoning of a purely thermodynamic character.

It is preferable to proceed as follows:

From the second law of Thermodynamics we deduce that in an adiabatic, reversible change the entropy of a system does not alter. Now acceleration of the velocity of a system in its frame of reference by conservative mechanical forces is certainly a reversible operation, and involves no transference of heat; hence the entropy of the system is not altered. From this we conclude that the entropy of a thermodynamic system

at rest in S′ is equal to its entropy in S. In short, *entropy is invariant.*

If now the system has its entropy altered from ϕ' to $\phi' + \delta\phi'$ by a reversible exchange of heat of amount $\delta Q'$ with a source at the same temperature T′, we know that

$$\delta\phi' = \delta Q'/T',$$

referring all measurements to S′.

Referring measurements to S, we have

$$\phi = \phi'$$
$$\phi + \delta\phi = \phi' + \delta\phi'$$

and therefore
$$\delta\phi = \delta\phi'.$$

But, according to now familiar ideas, the increase of energy (when measured in S) in the system is δQ where

$$\delta Q = \alpha\delta Q'.$$

At this point it is extremely easy to fall into a serious error by writing

$$\delta\phi = \delta Q/T,$$

and so obtaining the false conclusion that

$$T = \alpha T'.$$

To be sure, the system acquires an increase of energy $\delta Q (= \alpha\delta Q')$ in S; but we must remember that this involves an increase in mass and therefore in momentum, since we postulate a uniform velocity u in S. Hence in S a mechanical force must be acting on the system equal to $u dQ/dt$. This force will do work in time δt equal to

$$u\delta t(u\,.\,dQ/dt)$$
$$= u^2\delta Q.$$

This will be part of the increase of energy, but being mechanical work must be subtracted from the total increase δQ, to obtain the *amount of heat transferred* from the source to the system. This amount is therefore equal to

$$\delta Q(1 - u^2)$$
$$= \delta Q'/\alpha.$$

Hence
$$\frac{\delta Q'}{\alpha T} = \delta\phi = \delta\phi' = \frac{\delta Q'}{T'}$$

and therefore
$$T' = \alpha T,$$

which is the correct result.

PART II.

GENERAL RELATIVITY.

CHAPTER VIII.

In the world of physical events, the conceptions of distance between points and interval between instants have been deprived of the property of possessing absolute and invariant values which appertained to them in the space and time of the older Physics. They are now components (variable with the frame of reference) of a single invariant magnitude, separation between events. The manner of compounding the separation from its component time and space elements has already been indicated, but a certain limitation has been implicitly imposed on the choice of frames of reference in employing this method of composition. We have seen the difficulty of bringing gravitational force as a magnitude obeying the Relativity test into our scheme, and so it would appear to be a necessary assumption to postulate that the Relativity Mechanics and Electromagnetic theory which have been developed in the previous chapters are valid only for frames of reference in which gravitational action is absent, i.e., in which the rectilinear motion due to inertia is the most obvious feature and curvilinear paths do not exist, unless produced by so-called mechanical actions such as pressures, pulls, impacts, etc., or by electromagnetic action. Now no such frames are available for us, unless we adopt the device indicated in the Introduction of attaching our axes to a minute portion of matter travelling in a natural orbit with the acceleration characteristic of its position with reference to the gravitating bodies which influence it; and even then gravitation is "removed" only for a minute portion of space around the origin, but not for a widely extended portion; at some distance from this minute particle natural motion would be curvilinear in this frame also. This is so because, referred to our ordinary frames, the accelerations characteristic of the place where the particle is, and of any other place at a finite distance from it, are in general different.

But although finitely-extended non-gravitational frames do not exist, it is clear that we can imitate closely many of the mechanical effects of gravitation by conceiving the movements

in such a frame as referred to a second frame moving in some non-uniform manner with respect to the former. For example, such a conception would reproduce perfectly the most striking feature of gravitational action, the non-dependence of the acceleration on the mass or nature of the accelerated body, provided there is no resisting medium. This suggests, therefore, that if we could develop a calculus which would generalise the method of transformation adopted in Chapter VI., and remove the restriction of linear equations of transformation, we should be in a position so to generalise the mathematical forms of the laws of Physics as to give them validity in gravitational frames of reference.

As an initial step it would be natural to determine the new expression for the separation between two events. We can illustrate this by a few examples. Thus, suppose we had a non-gravitational or " Galilean " frame S′, and conceive another frame S to be rotating uniformly with respect to S′ around the axis of z' with a uniform angular velocity ω. The familiar " joy-wheel " is an example of such a frame in which all the people sharing in the general rotatory motion of the wheel experience a centrifugal acceleration which carries them off the wheel precisely as if each one were subject to a repulsive gravitational force from the axis proportional to the square of his distance from the axis, and also (the most important point of all) to his *mass*. We could adopt as equations of transformation (approximately true at all events)

$$\begin{aligned} x' &= x \cos \omega t - y \sin \omega t \\ y' &= x \sin \omega t + y \cos \omega t \\ z' &= z \\ t' &= t. \end{aligned}$$

So if two events occurred in S′ whose co-ordinates are (x', y', z', t') and $(x' + \delta x', \ldots t' + \delta t')$, their co-ordinates in S would be (x, y, z, t) and $(x + \delta x, \ldots t + \delta t)$, where

$$\begin{aligned} \delta x' &= \delta x \cos \omega t - \delta y \sin \omega t - \omega(x \sin \omega t + y \cos \omega t)\delta t \\ \delta y' &= \delta x \sin \omega t + \delta y \cos \omega t + \omega(x \cos \omega t - y \sin \omega t)\delta t \\ \delta z' &= \delta z \\ \delta t' &= \delta t. \end{aligned}$$

Hence the separation between the two events as measured in S would be given by calculating $\delta t'^2 - \delta x'^2 - \delta y'^2 - \delta z'^2$ in terms of $\delta x, \ldots \delta t$. This turns out to be

$$\{[1-\omega^2(x^2+y^2)]\delta t^2-\delta x^2-\delta y^2-\delta z^2+2\omega y\delta x\delta t-2\omega x\delta y\delta t\}^{\frac{1}{2}} \quad (1)$$

The separation between two events finitely separated would be obtained by an integration along a world line joining the events. As another illustration, consider the case where the inertial motion in a frame S′ is referred to a frame S which has a movement of uniform *rest* acceleration with regard to S′. The equations of transformation are

$$\begin{aligned} x' &= x + g^{-1} \cosh kt \\ y' &= y\,;\ z' = z \\ t' &= g^{-1} \sinh kt, \end{aligned}$$

for these equations give

$$(x' - x)^2 - t'^2 = (1/g)^2,$$

and thus a point fixed in S (i.e., $x =$ constant) has a uniform rest acceleration in S of amount g. (See equation (12) of Chapter II.)

The differentials are related by

$$\begin{aligned} \delta x' &= \delta x + kg^{-1} \sinh kt \,.\, \delta t \\ \delta y' &= \delta y\,;\ \delta z' = \delta z \\ \delta t' &= kg^{-1} \cosh kt \,.\, \delta t, \end{aligned}$$

and so the square of an element of separation is

$$k^2 g^{-2} \delta t^2 - \delta x^2 - \delta y^2 - \delta z^2 - 2kg^{-1} \sinh kt \,.\, \delta x \delta t \quad . \quad (2)$$

Another interesting example, due to Lorentz, consists in transforming from the inertial frame S′ to S by means of the equations,

$$\begin{aligned} x' &= x \cosh kt \\ y' &= y\,;\ z' = z \\ t' &= x \sinh kt, \end{aligned}$$

which implies that a point fixed in S has a rest acceleration in S′ whose value varies inversely as its x co-ordinates in S; for

$$x'^2 - t'^2 = x^2.$$

The differential relations are

$$\left.\begin{aligned} \delta x' &= kx \sinh kt \,.\, \delta t + \cosh kt \,.\, \delta x \\ \delta y' &= \delta y\,;\ \delta z' = \delta z \\ \delta t' &= kx \cosh kt \,.\, \delta t + \sinh kt \,.\, \delta x \end{aligned}\right\} \quad . \quad . \quad (3)$$

and thus the square of an element of separation is

$$k^2 x^2 \delta t^2 - \delta x^2 - \delta y^2 - \delta z^2 \quad . \quad . \quad . \quad (4)$$

In general, whatever the nature of the relative motion of S to S′, the co-ordinates of an event in S would be connected with those of the same event in S′ by relations such as

$$\begin{aligned} x' &= f_1(x, y, z, t) \\ y' &= f_2(x, y, z, t) \\ z' &= f_3(x, y, z, t) \\ t' &= f_4(x, y, z, t) \end{aligned}$$

where f_1, f_2, f_3, f_4 would be functions depending on the relative motion of S and S′; and so the "geometric" field of gravitation existing in S could be said to depend on these four functions. The differential relations would be

$$\left.\begin{aligned} \delta x' &= a_{11}\delta x + a_{12}\delta y + a_{13}\delta z + a_{14}\delta t \\ & \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ & \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \delta t' &= a_{41}\delta x + a_{42}\delta y + a_{43}\delta z + a_{44}\delta t \end{aligned}\right\} \quad . \quad . \quad (5)$$

where

$$a_{11} = \partial f_1/\partial x, \text{ etc., } a_{44} = \partial f_4/\partial t \quad . \quad . \quad (6)$$

The element of separation in S would be the square root of

$$\begin{aligned} & g_{11}\delta x^2 + g_{22}\delta y^2 + g_{33}\delta z^2 + g_{44}\delta t^2 \\ & + 2g_{12}\delta x\delta y + 2g_{13}\delta x\delta z + 2g_{14}\delta x\delta t \\ & + 2g_{23}\delta y\delta z + 2g_{24}\delta y\delta t + 2g_{34}\delta z\delta t \end{aligned} \quad . \quad . \quad (7)$$

where g_{11}, etc., g_{44} are ten functions given by

$$\left.\begin{aligned} g_{11} &= a_{41}^2 - a_{11}^2 - a_{21}^2 - a_{31}^2 \\ & \quad \cdot \quad \cdot \quad \cdot \\ g_{44} &= a_{44}^2 - a_{14}^2 - a_{24}^2 - a_{34}^2 \\ g_{12} &= a_{41}a_{42} - a_{11}a_{12} - a_{21}a_{22} - a_{31}a_{32} \end{aligned}\right\} \quad . \quad (8)$$

The change involved, therefore, in passing from an inertial frame of reference to one in which quasi-gravitational motion exists can be expressed by saying that the mathematical form for the square of an element of separation is a general quadratic function of the co-ordinate differentials instead of the special form with which we have been familiarised in the restricted theory, and that the ten coefficients are functions characteristic of the geometric field of gravitation.

Now the hypothesis which Einstein makes is that there is an equivalence between the mathematical forms of physical laws in our actual frames of reference and in the accelerated frames introduced above. True, the equivalence is not perfect; indeed, Einstein's law of gravitation can be considered

as a definite indication of the point where the equivalence fails. This, however, will arise at a later stage; we shall for the present accept this " Principle of Equivalence " and proceed to develop its consequences when linked with the Principle of General Relativity, which require us to represent the laws of nature in such general mathematical forms that any transformation of co-ordinates, however arbitrary, will not alter the form of these laws. As may naturally be expected, the laws in question are the laws of general dynamics, of electromagnetic theory, and of gravitation itself.

As an illustration of this combination of Relativity with Equivalence, let us consider the " natural " path of a " free " particle in S, i.e., a particle which has a uniform motion in S'. We shall use in future the symbol s for the invariant separation between two events, so that *

$$\begin{aligned}\delta s^2 &= \delta t'^2 - \delta x'^2 - \delta y'^2 - \delta z'^2\\ &= g_{11}\delta x^2 + \ . \ . \ . + g_{44}\delta t^2\\ &\quad + 2g_{23}\delta y\delta z + \ . \ . \ . + 2g_{34}\delta z\delta t.\end{aligned}$$

In the frame S' the natural rectilinear motion of the particle is given by the equations of Chapter III., viz.,

$$\ddot{x}' = 0, \ddot{y}' = 0, \ddot{z}' = 0, \ddot{t}' = 0 \quad . \quad . \quad (9)$$

where the dots refer to total differentiation with respect to s.

A very convenient method of transforming these equations to the variables x, y, z, t is that due to Lagrange, with which every student of general dynamics will be familiar. If we write $2L$ for the function,

$$g_{11}\dot{x}^2 + \ . \ . \ . + g_{44}\dot{t}^2 + 2g_{12}\dot{x}\dot{y} + \ . \ . \ . + 2g_{34}\dot{z}\dot{t},$$

it can be shown that the transformed equations are

$$\left.\begin{aligned}d(\partial L/\partial\dot{x})/ds - \partial L/\partial x &= 0\\ d(\partial L/\partial\dot{y})/ds - \partial L/\partial y &= 0\\ d(\partial L/\partial\dot{z})/ds - \partial L/\partial z &= 0\\ d(\partial L/\partial\dot{t})/ds - \partial L/\partial t &= 0\end{aligned}\right\} \quad . \quad . \quad (10)$$

* In the preceding chapters we have used the symbol δs^2 to represent $(\delta x'^2 + \delta y'^2 + \delta z'^2 - \delta t'^2)$, since we were employing imaginary time in order to introduce symmetry into the mathematical equations, so that δs was really an imaginary quantity. The necessity for this has, however, disappeared, as we shall see, and so this change in symbolism has been made. In fact, δs is the $\delta\tau$ of the simple Lorentz equations in the earlier chapters.

These, then, are the general equations of the "natural" paths of a particle in the frame S. It is obvious to anyone familiar with general analysis that they satisfy the test of general Relativity, but the point will be precisely demonstrated later. The Principle of Equivalence is invoked if we postulate that these also are the forms of the equations of a free particle's path in any natural frame of reference.

Another illustration concerns the path of a ray of light in S. In S′ it is, of course, straight with a uniform velocity (the unit); its differential equation is

$$\delta t'^2 - \delta x'^2 - \delta y'^2 - \delta z'^2 = 0,$$

or

$$\delta s = 0.$$

So in S it is in general curved, and its differential equation is

$$g_{11}\delta x^2 + \ldots + g_{44}\delta t^2 + 2g_{12}\delta x\delta y + \ldots + 2g_{34}\delta x\delta t = 0.$$

For example, taking the form (4) above, we have

$$k^2x^2\delta t^2 - \delta x^2 - \delta y^2 - \delta z^2 = 0,$$

and so the velocity of light at a "level" x in the frame S would be kx in any direction. This variation in velocity involves curvature of path, as is clear from other considerations.*

* This particular illustration, due to Lorentz, has some points of special interest. Thus the equation of a "falling" body in S (i.e., one fixed in S is $x = a \operatorname{sech} kt$ (where a is a constant, viz., its level at $t = 0$) so that

$$\begin{aligned} dx/dt &= -ka \operatorname{sech} kt \tanh kt \\ &= -kx \tanh kt. \end{aligned}$$

This approaches kx, the velocity of light at x as t approaches infinity, and never exceeds the velocity of light at the particular place through which it is passing at an instant.

Similarly its acceleration d^2x/dt^2 can be worked out to be $k^2x(1 - 2x^2/a^2)$, which suffers a change of direction when the body reaches the level $x = a/\sqrt{2}$, causing the body asymptotically to approach the level $x = 0$ with a velocity asymptotically diminishing to zero.

A further interesting feature can be elicited if the equations (3) are written

$$\begin{aligned} \delta x' &= a(\delta x - u\delta t) \\ \delta y' &= \delta y \,;\ \delta z' = \delta z \\ c'\delta t' &= a(c\delta t - u/c \,.\, \delta x), \end{aligned}$$

where

$$\begin{aligned} a &= \cosh kt \\ au &= -\sinh kt \\ c &= kx \\ c' &= 1, \end{aligned}$$

A third illustration concerns the possibility of a change in the periodic time of some natural occurrence, such as the vibration of an atomic radiating mechanism.

Consider two identical atoms situate at different places (x_1, y_1, z_1), (x_2, y_2, z_2), in the frame S. The proper time of a vibration at the first place would be $(g_{44})_1^{\frac{1}{2}}\delta t_1$ (since $\delta x_1 = 0 = \delta y_1 = \delta z_1$), at the second $(g_{44})_2^{\frac{1}{2}}\delta t_2$.

Now if we make the physical assumption that the identical properties of the atoms involve an equality of proper time, it is clear that δt_1 is not equal to δt_2 if $(g_{44})_1$ is not equal to $(g_{44})_2$, which is generally a fact.

The principle of Equivalence will, then, lead us to conclude that a field of natural gravitation will produce a curvature in the path of a beam of light, and also have an effect on the positions of spectral lines obtained from a source of light situated in it.

Passing to more general considerations, we are now in a position to grasp the nature of the problem facing us. We have to develop a method which enables us to test with the minimum of effort whether a given differential equation preserves the same form or not after a general transformation of variables subject to the invariance of a certain quadratic differential form, and we have to determine in the light of experiment and observation what are the conditions which must be satisfied in any natural frame of reference by the ten coefficients of that

and, therefore,

$$a = (1 - u^2/c^2)^{-\frac{1}{2}}.$$

Reciprocally we have

$$\begin{aligned} \delta x &= a(\delta x' - u'\delta t') \\ \delta y &= \delta y' ; \ \delta z = \delta z' \\ c\delta t &= a(c'\delta t' - u'/c' \, . \, \delta x') \\ u'/c' &= u/c. \end{aligned}$$

This is, in fact, a simple Lorentz transformation generalised to suit the assumption that the S and S′ observers correlate their measures of time by using c and c' as their velocities of light. Of course, the transformation could only be employed by two groups of S and S′ observers who happen to be adjacent, and only for occurrences in their immediate neighbourhood ; that is a condition imposed by the fact that the equations are in terms of differentials and are only true in so far as we can neglect the squares of the differentials. They are, however, suggestive of the momentarily existing relations between the measurements of length and time made by two groups of observers who pass one another in a given locality. A body fixed in one frame will be shorter in the ratio $a : 1$ to observers in the other. An occurrence at a locality fixed in one frame will take a longer time in the ratio $1 : a$ for observers in the other.

differential form ; in short, we have to develop a more general Tensor Analysis than that expounded in Chapter VI., and also discover a law of gravitation.

It will prove convenient once more to adopt the symbols x_1, x_2, x_3, x_4 instead of x, y, z, t, bearing in mind, however, that x_4 represents " real " time in future and not " imaginary."

VECTORS.

We have a set of co-ordinates (x_1, x_2, x_3, x_4), and the invariant separation between two neighbouring events is given by

$$\delta s^2 = g_{11}\delta x_1^2 + 2g_{12}\delta x_1\delta x_2 + \ldots + g_{44}\delta x_4^2 \qquad . \quad (11)$$

Let us transform to other axes of space and time involving new co-ordinates, which we will denote by (x_1', x_2', x_3', x_4').*

There is a linear relation between the differentials, viz.,

$$\left.\begin{array}{l} \delta x_1' = a_{11}\delta x_1 + \ldots + a_{14}\delta x_4 \\ \quad . \qquad . \qquad . \\ \quad . \qquad . \qquad . \\ \delta x_4' = a_{41}\delta x_1 + \ldots + a_{44}\delta x_4 \end{array}\right\} \quad . \quad . \quad (12)$$

where $a_{11}, \ldots, a_{44}$ are in general functions of x_1, x_2, x_3, x_4. (*N.B.*—It is not in general true that $a_{\lambda\mu} = a_{\mu\lambda}$.)

We can write these in contracted form as before, thus :

$$\delta x_\lambda' = a_{\lambda\alpha}\delta x_\alpha \qquad . \quad . \quad . \quad (12A)$$

Now in these new co-ordinates δs^2 will be equal to

$$g_{11}'\delta x_1'^2 + 2g_{12}'\delta x_1'\delta x_2' + \ldots + g_{44}'\delta x_4'^2,$$

where $g_{11}', \ldots, g_{44}'$ are coefficients which are functions of x_1', x_2', x_3', x_4'. But by the invariancy of δs^2 this must be equal to the right-hand side of (11). Hence it is not difficult to work out that

$$\begin{aligned} g_{11} = a_{11}^2 g_{11}' &+ a_{21}^2 g_{22}' + a_{31}^2 g_{33}' + a_{41}^2 g_{44}' \\ &+ 2a_{11}a_{21}g_{12}' + 2a_{11}a_{31}g_{13}' + 2a_{11}a_{41}g_{14}' \\ &\qquad + 2a_{21}a_{31}g_{23}' + 2a_{21}a_{41}g_{24}' \\ &\qquad\qquad + 2a_{31}a_{41}g_{34}', \end{aligned}$$

and nine other similar equations.

* Earlier in this chapter we were using accented co-ordinates to refer to another point or event in the same system of co-ordinates. The reader should be on his guard to avoid confusion between that and the present use of accented co-ordinates as referring to the co-ordinates of the same event in another system of co-ordinates.

The one just written can be succinctly expressed thus:

$$g_{11} = a_{\alpha 1}a_{\beta 1}g_{\alpha\beta}',$$

employing dummy suffixes, remembering that $g_{\lambda\mu} = g_{\mu\lambda}$ (but not *necessarily* that $a_{\alpha\beta} = a_{\beta\alpha}$).

Indeed, any one of the ten equations can be written

$$g_{\lambda\mu} = a_{\alpha\lambda}a_{\beta\mu}g_{\alpha\beta}' \quad . \quad . \quad . \quad . \quad (13)$$

Nominally, there are sixteen of these equations and sixteen terms on the right-hand side of each. Really there are, owing to the equality of $g_{\alpha\beta}$ and $g_{\beta\alpha}$, only ten equations, and the terms on the right-hand side of each can be reduced to ten.

Consider now a scalar function of the unaccented co-ordinates of an event, $\phi(x_1, x_2, x_3, x_4)$. Expressed in terms of the accented co-ordinates of the same event, this is $\psi(x_1', x_2', x_3', x_4')$, and

$$\phi(x_1, x_2, x_3, x_4) = \psi(x_1', x_2', x_3', x_4').$$

For a neighbouring event these functions are $\phi + \delta\phi$ and $\psi + \delta\psi$ where

$$\delta\phi = \partial\phi/\partial x_\alpha \,.\, \delta x_\alpha$$

and

$$\delta\psi = \partial\psi/\partial x_\alpha' \,.\, \delta x_\alpha'.$$

These are also equal since

$$\phi + \delta\phi = \psi + \delta\psi.$$

In consequence of equations (12),

$$\partial\phi/\partial x_\alpha \,.\, \delta x_\alpha = \partial\psi/\partial x_\alpha' \,.\, (a_{\alpha 1}\delta x_1 + a_{\alpha 2}\delta x_2 + a_{\alpha 3}\delta x_3 + a_{\alpha 4}\delta x_4)$$

(there are really four terms on the left-hand and sixteen terms on the right-hand side). Hence, on writing out in full and equating coefficients of δx_1, of δx_2, of δx_3, of δx_4, we get four equations, which, written out in full, are:

$$\left.\begin{aligned}
\partial\phi/\partial x_1 &= a_{11}\partial\psi/\partial x_1' + a_{21}\partial\psi/\partial x_2' + a_{31}\partial\psi/\partial x_3' + a_{41}\partial\psi/\partial x_4' \\
\partial\phi/\partial x_2 &= a_{12}\partial\psi/\partial x_1' + a_{22}\partial\psi/\partial x_2' + a_{32}\partial\psi/\partial x_3' + a_{42}\partial\psi/\partial x_4' \\
\partial\phi/\partial x_3 &= a_{13}\partial\psi/\partial x_1' + a_{23}\partial\psi/\partial x_2' + a_{33}\partial\psi/\partial x_3' + a_{43}\partial\psi/\partial x_4' \\
\partial\phi/\partial x_4 &= a_{14}\partial\psi/\partial x_1' + a_{24}\partial\psi/\partial x_2' + a_{34}\partial\psi/\partial x_3' + a_{44}\partial\psi/\partial x_4'
\end{aligned}\right\} \quad (14)$$

Our immediate purpose is, however, to express the functions $\partial\psi/\partial x_\lambda'$ in terms of the functions $\partial\phi/\partial x_\lambda$, i.e., to solve equations (14), regarded as linear simultaneous equations in $\partial\psi/\partial x_\lambda'$. To this end and to avoid continual digression later, we shall now collect those properties of the determinant

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$$

which will be required.

This determinant we denote by $|a|$. To any constituent corresponds a co-factor, and we employ the symbol $b_{\lambda\mu}$ to denote the result of dividing the co-factor of $a_{\lambda\mu}$ by the determinant $|a|$. There are sixteen of these symbols, and we can also construct a determinant with them; we denote this determinant by $|b|$.

The following theorems can then be proved to be true by the theory of determinants. They are generalisations of the theorems in Chapter V., where certain restrictions were put upon the a coefficients but are now removed.

THEOREM I.

$$\begin{aligned} a_{\alpha\lambda}b_{\alpha\mu} &= 1 \text{ if } \lambda = \mu \\ &= 0 \text{ if } \lambda \neq \mu, \end{aligned}$$

and

$$\begin{aligned} a_{\lambda\alpha}b_{\mu\alpha} &= 1 \text{ if } \lambda = \mu \\ &= 0 \text{ if } \lambda \neq \mu. \end{aligned}$$

(This is the analogue of equations (2), (3), (4), and (5) in Chapter V.)

THEOREM II.

$$|a| \times |b| = 1.$$

(This is the analogue of Theorem I. of Chapter V.)

THEOREM III.

The co-factor of any constituent of the b-determinant is equal to the corresponding constituent of the a-determinant divided by $|a|$, and the co-factor of any constituent of the a-determinant is equal to the corresponding constituent of the b-determinant divided by $|b|$. E.g.,

$$\begin{vmatrix} b_{22} & b_{23} & b_{24} \\ b_{32} & b_{33} & b_{34} \\ b_{42} & b_{43} & b_{44} \end{vmatrix} = a_{11}/|a|.$$

(Analogous to Theorem II. of Chapter V.)

THEOREM IV.

The co-factor of any second minor of the b-determinant is equal to the corresponding second minor of the a-determinant divided by $|a|$ (and similarly if a and b are interchanged). E.g.,

$$\begin{vmatrix} b_{33} & b_{34} \\ b_{43} & b_{44} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} / |a|.$$

(Analogous to Theorem III. of Chapter V.)

Reverting now to equations (14) above, their solution gives

$$\partial\psi/\partial x_1' = b_{11}\partial\phi/\partial x_1 + b_{12}\partial\phi/\partial x_2 + b_{13}\partial\phi/\partial x_3 + b_{14}\partial\phi/\partial x_4 \quad (15)$$

and three similar equations, or

$$\partial\psi/\partial x_\lambda' = b_{\lambda\alpha}\partial\phi/\partial x_\alpha \quad . \quad . \quad . \quad (15\text{A})$$

Looking at the transformation equations (12) and (15), we see that we have now to deal with two types of vectors or tensors of the first order. First, there are vectors which transform like $\partial\phi/\partial x_\lambda$, i.e., like the gradient of a scalar function; this type is called "covariant." We denote such vectors by capital letters with the indicating digit written as a suffix proper; so that to say that A_1, A_2, A_3, A_4 are the components of a covariant vector in the unaccented co-ordinates is to say that the components of the same vector in the accented co-ordinates are related to the former as follows:—

$$A_1' = b_{11}A_1 + b_{12}A_2 + b_{13}A_3 + b_{14}A_4 \quad . \quad (16)$$

and three similar equations;

or
$$A_\lambda' = b_{\lambda\alpha}A_\alpha \quad . \quad . \quad . \quad . \quad (16\text{A})$$

Secondly, there is the type of vector which transforms like the element of a "world line," as in equations (12). This type is called "contravariant," and we denote such vectors by capital letters with the indicating digit written as an affix on the right-hand side; so that if A^1, A^2, A^3, A^4 are the components of a contravariant tensor, then

$$A'^1 = a_{11}A^1 + a_{12}A^2 + a_{13}A^3 + a_{14}A^4 \quad . \quad . \quad (17)$$

and three similar equations; or

$$A'^\lambda = a_{\lambda\alpha}A^\alpha \quad . \quad . \quad . \quad . \quad (17\text{A})$$

(Of course, when the restrictions of Chapter V. are imposed on the coefficients, it is easily seen that $a_{\lambda\mu} = b_{\lambda\mu}$, and both types of vectors merge into one type.)

It is easily proved from (16) and (17) that

$$A_\lambda = a_{\alpha\lambda} A_\alpha' \quad . \quad . \quad . \quad . \quad (18)$$

and
$$A^\lambda = b_{\alpha\lambda} A'^\alpha \quad . \quad . \quad . \quad . \quad (19)$$

It must be borne in mind that the a and b coefficients are not constants, but are in general functions of the co-ordinates.*

It is easily proved by equations (16) and (17), using Theorem I., that if A_λ be a covariant vector and B^λ a contravariant vector,

$$A_\alpha B'^\alpha = AB'^\alpha_\alpha;$$

in other words, $A_\alpha B^\alpha$ is invariant.

(It is necessary to bear in mind that this is not true for products of the components of two covariant or two contravariant vectors, i.e., $A_\alpha B_\alpha$ is *not* invariant, nor $A^\alpha B^\alpha$, because it is *not* in general true that $a_{\alpha\lambda} a_{\alpha\mu} = 1$ if $\lambda = \mu$, and equal to zero if $\lambda \neq \mu$, and similarly for the b coefficients.)

The converse of this proposition on invariancy is also true. Let A_1, A_2, A_3, A_4 be four quantities such that after transformation they become A_1', A_2', A_3', A_4'. Also let B^λ be *any arbitrary* contravariant vector. If after transformation it is true that

$$A_\alpha' B'^\alpha = A_\alpha B^\alpha,$$

then the A_λ functions constitute a covariant vector.

For, by hypothesis,

$$A_\alpha' B'^\alpha = A_\beta B^\beta$$

and $B^\lambda = b_{\alpha\lambda} B'^\alpha$, using equations (19).

Hence $A_\alpha' B_\alpha' = A_\beta b_{\alpha\beta} B^{\alpha\prime}$ (there are 16 terms on the right-hand side),

or $(A_\alpha' - b_{\alpha\beta} A_\beta) B'^\alpha = 0$.

Hence as B^λ is an arbitrary vector, the individual multipliers of B'^1, B'^2, B'^3, B'^4 in the left-hand expression must be zero. So

$$A_\lambda' = b_{\lambda\beta} A_\beta,$$

and from equation (16) we see that A_λ is a covariant vector.

A similar converse theorem is true if we interchange the words " contravariant " and " covariant " above.

* For the origin of the terms " covariant " and " contravariant," see the note at the end of the chapter.

TENSORS OF THE SECOND ORDER.

If A_λ and B_λ are two covariant vectors, and if we write down the sixteen quantities $A_\lambda B_\mu$, it is easy to see that on transformation we have

$$A_\lambda' B_\mu' = b_{\lambda\alpha} b_{\mu\beta} A_\alpha B_\beta,$$

there being sixteen such equations and sixteen terms on each right-hand side.

Any set of sixteen quantities which transform in this fashion constitute a tensor of the second order, and we denote such quantities by using double suffixes, as $A_{\lambda\mu}$, $B_{\lambda\mu}$, etc., so that, after transformation,

$$A_{\lambda\mu}' = b_{\lambda\alpha} b_{\mu\beta} A_{\alpha\beta}, \quad . \quad . \quad . \quad (20)$$

and as a consequence of this

$$A_{\lambda\mu} = a_{\alpha\lambda} a_{\beta\mu} A_{\alpha\beta}' \quad . \quad . \quad . \quad (21)$$

(See equations (18) and (19).)

Similarly a contravariant tensor of the second order is defined by sixteen components $A^{\lambda\mu}$, which satisfy the transformation relation

$$A'^{\lambda\mu} = a_{\lambda\alpha} a_{\mu\beta} A^{\alpha\beta}, \quad . \quad . \quad . \quad (22)$$

involving also

$$A^{\lambda\mu} = b_{\alpha\lambda} b_{\beta\mu} A'^{\alpha\beta} \quad . \quad . \quad . \quad . \quad (23)$$

The tensors $A_\lambda B_\mu$ and $A^\lambda B^\mu$ are referred to as the "outer" products of the two vectors A_λ and B_μ, or A^λ and B^μ. Such products form a special class of tensors of the second order, the general tensor not being expressible as such a product.

There is a third type of second order tensor called "mixed." Its transformation equation is similar to that of the "mixed" product $A_\lambda B^\mu$. A group of sixteen quantities forming a mixed tensor is denoted by such symbols as $A_\lambda{}^\mu$, $B_\lambda{}^\mu$, etc.; being covariant with respect to the suffix λ, and contravariant to the affix μ, its transformation equation is

$$A_\lambda'{}^\mu = b_{\lambda\alpha} a_{\mu\beta} A_\alpha{}^\beta, \quad . \quad . \quad . \quad . \quad (24)$$

involving also

$$A_\lambda{}^\mu = a_{\alpha\lambda} b_{\beta\mu} A_\alpha'{}^\beta \quad . \quad . \quad . \quad . \quad (25)$$

By Theorem I. it is easy to see that the sum of the four components of $A_\lambda{}^\mu$, which have suffix and affix equal, is invariant. In fact,

$$A_\alpha'{}^\alpha = A_\alpha{}^\alpha.$$

This is an example of a process called "contraction" of mixed tensors which will presently be explained more fully in connection with tensors of still higher orders. Indeed, the invariance of $A_\alpha B^\alpha$, already established, is an example of the contraction of a mixed tensor product.

A tensor of the second order whose components satisfy the conditions $A_{\lambda\mu} = A_{\mu\lambda}$, or $A^{\lambda\mu} = A^{\mu\lambda}$, is called a "symmetric" tensor, covariant or contravariant, as the case may be. If the conditions $A_{\lambda\mu} = - A_{\mu\lambda}$, or $A^{\lambda\mu} = - A^{\mu\lambda}$, are satisfied (involving $A_{11} = A_{22} = A_{33} = A_{44} = 0$, or $A^{11} = 0 =$ etc.), the tensor is called "antisymmetric," and precisely as in Chapter V., can be considered as a "six-vector" (covariant or contravariant).

A reference to equations (13) and (21) shows that the g-coefficients constitute a symmetric covariant tensor of the second order. This tensor, on account of the great importance of the expression for the invariant δs^2, is generally called the "fundamental" covariant tensor. In this case there is a slight departure from the usual notation, the suffix being attached to a small letter instead of to a capital.

Let us consider two second order tensors $A_{\lambda\mu}$ and $B^{\lambda\mu}$, covariant and contravariant respectively; we can show that the sum of the sixteen terms $A_{\alpha\beta}B^{\alpha\beta}$ is invariant. For by equations (20) and (22),

$$A_{\alpha\beta}'B'^{\alpha\beta} = a_{\gamma\alpha}a_{\delta\beta}B^{\gamma\delta}b_{\gamma\alpha}b_{\delta\beta}A_{\gamma\delta}.$$

There are 256 terms on the right-hand side. Fixing for the moment on definite values of γ and δ (i.e., considering sixteen of these terms alone), it is not difficult to see that in view of Theorem I.

$$\begin{aligned} &\sum_{\alpha\beta}(a_{\gamma\alpha}b_{\gamma\alpha}a_{\delta\beta}b_{\delta\beta}) \text{ (16 terms)} \\ &= (\sum_{\alpha} a_{\gamma\alpha}b_{\gamma\alpha})(\sum_{\beta} a_{\delta\beta}b_{\delta\beta}) \text{ (each bracket contains 4 terms)} \\ &= 1 \times 1 \\ &= 1. \end{aligned}$$

Hence the coefficient of each of the terms $A_{\gamma\delta}B^{\gamma\delta}$ on the right-hand side is unity. Hence $A_{\alpha\beta}B^{\alpha\beta}$ is invariant. (This is a further example of contraction of a mixed product.)

It is somewhat tedious but not difficult to prove the converse of this result, just as we did for vectors. If $A_{\lambda\mu}$ is a group of sixteen quantities which transform in such a way that $A_{\alpha\beta}B^{\alpha\beta} = A_{\alpha\beta}'B'^{\alpha\beta}$, where $B^{\lambda\mu}$ is *any arbitrary* contravariant tensor of the second order, then $A_{\lambda\mu}$ is a covariant tensor of the second

order. A similar theorem can be proved when "covariant" and "contravariant" are interchanged.

This result enables us to discover a *fundamental* contravariant tensor, for if we take the co-factors of the $g_{\lambda\mu}$ in the determinant $|g|$ and divide each of them by the value of $|g|$, the sixteen quotients form a (symmetric) contravariant tensor. Denoting them (in anticipation) by the symbols $g^{\lambda\mu}$, we know by the theory of determinants that

$$g_{11}g^{11} + g_{12}g^{12} + g_{13}g^{13} + g_{14}g^{14} = 1$$

and three similar equations ; or

$$\sum_{\alpha} g_{\lambda\alpha}g^{\lambda\alpha} = 1,$$

which proves the contravariancy of $g^{\lambda\mu}$ in the light of what we have just demonstrated.

Indeed,

$$\left.\begin{aligned} g_{\lambda\alpha}g^{\mu\alpha} &= 1 \text{ if } \lambda = \mu \\ &= 0 \text{ if } \lambda \neq \mu \end{aligned}\right\} \quad . \quad . \quad . \quad (26)$$

It can also be shown that the product of the determinant of the $g_{\lambda\mu}$ functions and that of the $g^{\lambda\mu}$ functions is unity, or

$$|g^{\lambda\mu}| = 1/|g|.$$

A relation exists between $|g|$ and $|g'|$ which is of considerable importance. It will be as well to deal with it now. If we write out the determinant $|g'|$ in full and the determinant $|a|$ of the coefficients $a_{\lambda\mu}$ in equations (12), and employ the rule for multiplying determinants, we obtain as their product the determinant

$$\begin{vmatrix} a_{\alpha 1}g_{\alpha 1}' & a_{\alpha 1}g_{\alpha 2}' & a_{\alpha 1}g_{\alpha 3}' & a_{\alpha 1}g_{\alpha 4}' \\ a_{\alpha 2}g_{\alpha 1}' & a_{\alpha 2}g_{\alpha 2}' & a_{\alpha 2}g_{\alpha 3}' & a_{\alpha 2}g_{\alpha 4}' \\ a_{\alpha 3}g_{\alpha 1}' & a_{\alpha 3}g_{\alpha 2}' & a_{\alpha 3}g_{\alpha 3}' & a_{\alpha 3}g_{\alpha 4}' \\ a_{\alpha 4}g_{\alpha 1}' & a_{\alpha 4}g_{\alpha 2}' & a_{\alpha 4}g_{\alpha 3}' & a_{\alpha 4}g_{\alpha 4}' \end{vmatrix}$$

where each constituent is the sum of four terms.

If we again multiply this determinant by $|a|$, we obtain a determinant of which each constituent is the sum of sixteen terms such as $a_{\alpha\lambda}a_{\beta\mu}g_{\alpha\beta}'$; in fact, it is the determinant $|g|$ as a reference to equations (13) will demonstrate.

Hence

$$|g| = |g'| \cdot |a|^2.$$

Presently we will see that the nature of the space-time

continuum is such that the g-determinants are everywhere essentially negative. Assuming this result for the time being, and dropping the bars at the side of the symbols for convenience, we see that the determinant of the a-coefficients used in the differential transformation from unaccented to accented co-ordinates is equal to $\surd(-g)/\surd(-g')$.

This result is of some importance in connection with the concept of four-dimensional volume. Taking a body at rest relative to an observer, and multiplying its ordinary three-dimensional volume by the interval of time between two events connected with that body, we obtain a "four-dimensional" volume. Referred to a frame in which it is moving, the body has a different volume and shape, but if the motion be uniform the shape and size are definite, and if we multiply this volume by the interval between the two events as measured in the new frame, we arrive at the same result as before. That we already know from the earlier chapters on the Lorentz transformation. We are, in fact, dealing with a cylindrical four-dimensional volume whose generating lines are the parallel world lines of the surface particles of the body. But if the motion in the frame be not uniform, the shape and size of the body are altering continuously, the world lines of the surface particles are not parallel, the three-dimensional section of the world tube of the body is varying, and in order to obtain the four-dimensional volume embracing the body at all instants between the events we must multiply its three-dimensional volume at an instant by a differential element of time in the frame and integrate. The final result has not the same value as before, but the relation connecting such four-dimensional volumes of a body between two events can be arrived at by applying a theorem due to Jacobi, which shows that in virtue of the transformation equations (12),

$$\iiiint dx_1'dx_2'dx_3'dx_4' = \iiiint |a| dx_1dx_2dx_3dx_4\,;$$

or, in view of the fact that at each definite point instant

$$|a| = \surd(-g)/\surd(-g'),$$

we can write it

$$\iiiint \surd(-g)dx_1dx_2dx_3dx_4 = \iiiint \surd(-g')dx_1'dx_2'dx_3'dx_4'.$$

This invariant integral is equal to the integral $\iiiint dxdydzdt$ in a Galilean frame, for in it $g = -1$, since its constituents have the values $g_{11} = g_{22} = g_{33} = -1$, $g_{44} = 1$, and the rest

zero. The integral $\iiiint dxdydzdt$ is the "natural" volume. Einstein points out that if g should vanish anywhere, an elementary small natural volume would correspond to a finite co-ordinate volume $\delta x_1\ \delta x_2\ \delta x_3\ \delta x_4$. This he assumes not to be the case anywhere in space-time accessible to us, making, in fact, a definite hypothesis concerning the physical nature of the continuum. If g does not vanish anywhere in any frame it preserves the same sign; and as it is negative in the restricted Relativity theory, its constant sign is negative, so that $\sqrt{(-g)}$ is a real quantity.

We can employ Theorem IV. to demonstrate the existence of a six-vector "reciprocal" to a given six-vector. Thus, if $A_{\lambda\mu}$ be a covariant antisymmetric tensor or six-vector, its transformation equations are

$$A_{\lambda\mu}' = \begin{vmatrix} b_{\lambda\alpha} & b_{\lambda\beta} \\ b_{\mu\alpha} & b_{\mu\beta} \end{vmatrix} A_{\alpha\beta},$$

where there are six terms on each right-hand side and the suffix $\alpha\beta$ goes through the sequence, 23, 31, 12, 14, 24, 34 in the summations. It is easily seen that the antisymmetry is preserved in the accented co-ordinates. If we use Theorem IV., we see that the thirty-six determinant coefficients can be replaced by coefficients which are second minors of $|a|$, each divided by $|a|$. Going through a procedure similar to that in Chapter VI., we find that $A_{14}/a, A_{24}/a, A_{34}/a, A_{23}/a, A_{31}/a, A_{12}/a$ transform into $A_{14}', A_{24}', A_{34}', A_{23}', A_{31}', A_{12}'$ by exactly the same equations as the components of a contravariant six-vector *in this order*, B^{23}, B^{31}, B^{12}, B^{14}, B^{24}, B^{34}, transform into B'^{23}, B'^{31}, B'^{12}, B'^{14}, B'^{24}, B'^{34}. But $|a| = \sqrt{(-g)}/\sqrt{(-g')}$. Hence we prove that if $A_{\lambda\mu}$ be a covariant six-vector, there exists a contravariant six-vector $B^{\lambda\mu}$, such that $A_{23} = \sqrt{(-g)}B^{14}$, etc., . . ., $A_{34} = \sqrt{(-g)}B^{12}$. $B^{\lambda\mu}$ is "reciprocal" to $A_{\lambda\mu}$.

Similarly, if $A^{\lambda\mu}$ be a contravariant six-vector, there exists a reciprocal covariant six-vector $B_{\lambda\mu}$, such that

$$A^{23} = B_{14}/\sqrt{(-g)},\ .\ .\ .,\ A^{34} = B_{12}/\sqrt{(-g)}.$$

Tensors of Higher Order: Outer and Inner Multiplication: Contraction.

We can proceed by very obvious generalisation to tensors of higher order than the second. Thus a covariant tensor of the third order consists of sixty-four components $A_{\lambda\mu\nu}$, such that the components in accented co-ordinates are related to

those in unaccented co-ordinates by the sixty-four equations

$$A_{\lambda\mu\nu}' = b_{\lambda\alpha}b_{\mu\beta}b_{\nu\gamma}A_{\alpha\beta\gamma},$$

each of which has sixty-four terms on the right-hand side, as α, β, γ each go through the sequence 1, 2, 3, 4 independently.

Particular cases of such tensors are the "outer" product of a covariant tensor of the first order (or vector) and one of the second, $A_{\lambda\mu}B_{\nu}$, or the outer product of three covariant vectors $A_{\lambda}B_{\mu}C_{\nu}$.

Similarly a contravariant tensor of the third order has its components in two systems connected by

$$A'^{\lambda\mu\nu} = a_{\lambda\alpha}a_{\mu\beta}a_{\nu\gamma}A^{\alpha\beta\gamma},$$

and special cases are $A^{\lambda\mu}B^{\nu}$ and $A^{\lambda}B^{\mu}C^{\nu}$.

As before, we can prove the invariance of

$$A_{\alpha\beta\gamma}B^{\alpha\beta\gamma},$$

which is the sum of sixty-four terms, and also derive a corresponding converse theorem. Here we have a further example of "contraction."

A mixed tensor of the third order may either be of the type $A_{\lambda\mu}{}^{\nu}$ or $A_{\lambda}{}^{\mu\nu}$, i.e., covariant with respect to two indexes, and contravariant to one, or *vice versa*. In the former case,

$$A'_{\lambda\mu}{}^{\nu} = b_{\lambda\alpha}b_{\mu\beta}a_{\nu\gamma}A_{\alpha\beta}{}^{\gamma};$$

in the latter,

$$A'_{\lambda}{}^{\mu\nu} = b_{\lambda\alpha}a_{\mu\beta}a_{\nu\gamma}A_{\alpha}{}^{\beta\gamma}.$$

Special examples are such mixed products as $A_{\lambda\mu}B^{\nu}$ or $A_{\lambda}B_{\mu}C_{\nu}$ and $A_{\lambda}B^{\mu\nu}$, or $A_{\lambda}B^{\mu}C^{\nu}$.

By a more general application of the process of contraction than we have hitherto used, we can derive vectors or tensors of the first order from mixed tensors of the third order. Suppose for example we consider $A_{\lambda\alpha}{}^{\alpha}$; this means the sum of four of the components of the mixed tensor $A_{\lambda\mu}{}^{\nu}$. There are four such sums, viz.,

$$\begin{aligned}
&A_{11}{}^{1} + A_{12}{}^{2} + A_{13}{}^{3} + A_{14}{}^{4}\\
&A_{21}{}^{1} + A_{22}{}^{2} + A_{23}{}^{3} + A_{24}{}^{4}\\
&A_{31}{}^{1} + A_{32}{}^{2} + A_{33}{}^{3} + A_{34}{}^{4}
\end{aligned}$$

and

$$A_{41}{}^{1} + A_{42}{}^{2} + A_{43}{}^{3} + A_{44}{}^{4}.$$

That is, we have selected sixteen of the components of $A_{\lambda\mu}{}^{\nu}$, which have the affix equal to the second suffix, and summed them in tetrads in a definite way. It is not difficult by

means of Theorem I. to show that these four quantities constitute a covariant vector ; or

$$A'_{\lambda\alpha}{}^{\alpha} = b_{\lambda\beta}A_{\beta\alpha}{}^{\alpha}.$$

Similarly we can " contract " the mixed tensor $A^{\lambda\mu}{}_{\nu}$ into a contravariant vector $A^{\lambda\alpha}{}_{\alpha}$, for which

$$A'^{\lambda\alpha}{}_{\alpha} = a_{\lambda\beta}A^{\beta\alpha}{}_{\alpha}.$$

When we apply this process to the mixed tensors obtained by outer multiplication of tensors, the results are termed " inner " products. Thus $A_{\lambda\alpha}B^{\alpha}$ in the " inner " product of $A_{\lambda\mu}$ and B^{ν}, and $A^{\lambda\alpha}B_{\alpha}$ of $A^{\lambda\mu}$ and B_{ν}.*

The invariant $A_{\alpha}B^{\alpha}$ is, of course, the inner product of the vectors A_{λ} and B^{μ}.

A tensor of the fourth order consists of 256 components. Its transformation equation if covariant is

$$A'_{\kappa\lambda\mu\nu} = b_{\kappa\alpha}b_{\lambda\beta}b_{\mu\gamma}b_{\nu\delta}A_{\alpha\beta\gamma\delta},$$

if contravariant

$$A'^{\kappa\lambda\mu\nu} = a_{\kappa\alpha}a_{\lambda\beta}a_{\mu\gamma}a_{\nu\delta}A^{\alpha\beta\gamma\delta},$$

if mixed,

$$A'_{\kappa\lambda\mu}{}^{\nu} = b_{\kappa\alpha}b_{\lambda\beta}b_{\mu\gamma}a_{\nu\delta}A_{\alpha\beta\gamma}{}^{\delta},$$

or

$$A'_{\kappa\lambda}{}^{\mu\nu} = b_{\kappa\alpha}b_{\lambda\beta}a_{\mu\gamma}a_{\nu\delta}A_{\alpha\beta}{}^{\gamma\delta},$$

or

$$A_{\kappa}{}^{\lambda\mu\nu} = b_{\kappa\alpha}a_{\lambda\beta}a_{\mu\gamma}a_{\nu\delta}A_{\alpha}{}^{\beta\gamma\delta}.$$

There are 256 terms on the right-hand side.

" Contraction " of $A_{\kappa\lambda\mu}{}^{\nu}$ gives $A_{\kappa\lambda\alpha}{}^{\alpha}$, i.e., involves a selection of sixty-four components of $A_{\kappa\lambda\mu}{}^{\nu}$, arranges them in groups of four and adding these tetrads, gives sixteen quantities which have the properties of a covariant tensor of the second order, i.e.,

$$A'_{\kappa\lambda\alpha}{}^{\alpha} = b_{\kappa\beta}b_{\lambda\gamma}A_{\beta\gamma\alpha}{}^{\alpha}.$$

Similarly $A^{\kappa\lambda\alpha}{}_{\alpha}$ is a contravariant tensor of the second order.

Taking $A_{\kappa\lambda}{}^{\mu\nu}$, we first contract it into $A_{\kappa\alpha}{}^{\mu\alpha}$, which is a mixed tensor of the second order, and if we apply the process of contraction once more we obtain the invariant $A_{\beta\alpha}{}^{\beta\alpha}$, of which we met an illustration above in the invariant quantity $A_{\alpha\beta}B^{\alpha\beta}$.

We have examples of contracted tensors in the " inner " products $A_{\lambda\mu\alpha}B^{\alpha}$ (covariant), $A^{\lambda\mu\alpha}B_{\alpha}$ (contravariant), etc.

The process of contraction can only be applied to mixed tensors ; it reduces the order of the tensor by two, and

* These are the analogues of the vector product [**A** . $\mathfrak{B}$] of a vector and a tensor of the second order in Chapter VI.

ultimately, after as many contractions as are possible, leaves a covariant or contravariant tensor or an invariant according as the number of suffixes is greater than, less than, or equal to the number of affixes.

A symmetric tensor of the third order only involves twenty numerically different components, for any six which involve the same suffixes or affixes (numerically different) in different orders are equal, and so twenty-four of the components reduce to four. Then $A_{112} = A_{211} = A_{121}$ and $A_{221} = A_{122} = A_{212}$, so that these six reduce to two and five other similar sets of six reduce to ten. There remain the four, A_{111}, A_{222}, A_{333}, A_{444}. Thus there are twenty in all.

If the tensor be antisymmetric there are only four numerically different components. All those involving at least two equal suffixes or affixes are zero; as regards the remaining twenty-four with different suffixes, we have, e.g.,

$$A_{123} = -A_{132} = A_{312} = -A_{321} = A_{231} = -A_{213}.$$

Hence the transformation equations in this case reduce to four in number, viz.,

$$A_{\lambda\mu\nu}' = \begin{vmatrix} b_{\lambda\alpha} & b_{\lambda\beta} & b_{\lambda\gamma} \\ b_{\mu\alpha} & b_{\mu\beta} & b_{\mu\gamma} \\ b_{\nu\alpha} & b_{\nu\beta} & b_{\nu\gamma} \end{vmatrix} A_{\alpha\beta\gamma} \text{ (four terms).}$$

where on the right-hand side $\alpha\beta\gamma$ goes through the sequence 234, 314, 124, 321 in the summation, and $\lambda\mu\nu$ is replaced by the same groups in succession. Remembering that $|a| = \sqrt{(-g)}/(-g')$, we can, by Theorem III., prove that if $A_{\lambda\mu\nu}$ is covariant and antisymmetric, there exists a contravariant vector B^λ such that $\sqrt{(-g)}B^1 = A_{234}$, $\sqrt{(-g)}B^2 = A_{314}$, $\sqrt{(-g)}B^3 = A_{124}$, $\sqrt{(-g)}B^4 = A_{321}$. Similarly if $A^{\lambda\mu\nu}$ is contravariant and antisymmetric, there exists a covariant vector B_λ, such that $B_1/\sqrt{(-g)} = A^{234}$, etc.

In a four-dimensional continuum there are no symmetric tensors higher than the fourth order, and an antisymmetric tensor of the fourth order has only one numerical value for all components such as A_{1234}, A_{1342}, etc., which are not zero.

Associated Tensors.

Considerable use is made of the process of contraction in connection with the fundamental tensors $g_{\lambda\mu}$ and $g^{\lambda\mu}$. Indeed, it reveals the tensor nature of these quantities very directly,

Thus $g_{\alpha\beta}\delta x_\alpha \delta x_\beta$ (i.e., $g_{11}\delta x_1{}^2 +$ etc.) is invariant (δs^2) ; so that, writing it in the form

$$(g_{1\alpha}\,\delta x_\alpha)\delta x_1 + (g_{2\alpha}\,\delta x_\alpha)\delta x_2 + (g_{3\alpha}\,\delta x_\alpha)\delta x_3 + (g_{4\alpha}\,\delta x_\alpha)\delta x_4,$$

and remembering that δx_1, δx_2, δx_3, δx_4 form a contravariant vector, we see that

$$g_{\lambda\alpha}\delta x_\alpha$$

is a covariant vector. Therefore, by the converse of one of the propositions above, this must be the mixed product of the contravariant vector δx_ν and a *covariant* tensor of the second order $g_{\lambda\mu}$.

Again, write

$$\delta y_\lambda = g_{\lambda\alpha}\,\delta x_\alpha,$$

so that δy_1, δy_2, δy_3, δy_4 constitute as stated a *covariant* vector. By solving we get

$$\delta x_\lambda = g^{\lambda\alpha}\delta y_\alpha.$$

The right-hand side is, in consequence, a contravariant vector, and must therefore be the inner product of the covariant vector δy_ν, and a *contravariant* tensor of the second order $g^{\lambda\mu}$.

From $g_{\lambda\mu}$ and $g^{\lambda\mu}$ we can form a mixed tensor of great importance by multiplication and one contraction. The outer product is $g_{\lambda\nu}\,g^{\mu\kappa}$, a mixed tensor of the fourth order. Contracting by putting $\nu = \kappa = \alpha$, we get $g_{\lambda\alpha}\,g^{\nu\alpha}$. We write this as $g_\lambda{}^\mu$, and by the theory of determinants we know that it is unity if $\lambda = \nu$, and zero if $\lambda \neq \nu$. Hence *in any co-ordinates* the mixed fundamental tensor $g_\lambda{}^\mu$ has the values given by the scheme

$$\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}$$

As a matter of fact, one can satisfy oneself very easily that this scheme transforms into itself by application of the mixed transformation equation

$$g'_\lambda{}^\mu = b_{\lambda\alpha}a_{\mu\beta}g_\alpha{}^\beta.$$

A further reduction gives $g_{\alpha\beta}g^{\alpha\beta}$, which is an invariant and has, as a matter of fact, the value 4 in all co-ordinates.

We can " associate " tensors of different order and character with a given tensor by use of the fundamental tensors.

For example, we can derive a contravariant vector from a covariant vector thus:

$$A^\lambda = g^{\lambda\alpha} A_\alpha$$

or a covariant from a contravariant thus:

$$A_\lambda = g_{\lambda\alpha} A^\alpha.$$

Likewise, for tensors of the second order, we can derive contravariant from covariant, or *vice versa*, or mixed tensors from either. E.g.,

$$\begin{aligned} A^{\lambda\mu} &= g^{\lambda\alpha}\, g^{\mu\beta} A_{\alpha\beta} \\ A_{\lambda\mu} &= g_{\lambda\alpha}\, g_{\mu\beta} A^{\alpha\beta} \\ A_\lambda{}^\mu &= g_{\lambda\alpha}\, A^{\mu\alpha} = g^{\mu\alpha}\, A_{\lambda\alpha}. \end{aligned}$$

The following theorem concerning the reciprocals of associated antisymmetric tensors of the second order will be of service later.

Let $A_{\lambda\mu}$ and $A^{\lambda\mu}$ be associated antisymmetric tensors, and let $B^{\lambda\mu}$ be the tensor reciprocal to $A_{\lambda\mu}$ and $B_{\lambda\mu}$ reciprocal to $A^{\lambda\mu}$, then the tensor associated with $B^{\lambda\mu}$ is the negative of $B_{\lambda\mu}$.

To prove this we must bear in mind that the cofactor of any second minor of the $g^{\lambda\mu}$ determinant is equal to the corresponding second minor of the $g_{\lambda\mu}$ determinant divided by g. This is, in fact, Theorem IV. above applied to g.

Now, taking, say, the 12-constituent of $A^{\lambda\mu}$, we have

$$\begin{aligned} A^{12} &= g^{1\alpha} g^{2\beta} A_{\alpha\beta} \text{ (12 terms)} \\ &= \begin{vmatrix} g^{1\alpha} g^{1\beta} \\ g^{2\alpha} g^{2\beta} \end{vmatrix} A_{\alpha\beta} \text{ (6 terms)} \end{aligned}$$

where, in the second line, $\alpha\beta$ has to go through the sequence 23, 31, 12, 14, 24, 34.

Hence

$$B_{34}/\sqrt{(-g)} = \begin{vmatrix} g_{3\gamma} g_{3\delta} \\ g_{4\gamma} g_{4\delta} \end{vmatrix} B^{\gamma\delta}/\sqrt{(-g)} \quad \text{(6 terms)}$$

(where $\gamma\delta$ corresponds to $\alpha\beta$, as 14 does to 23, or 24 to 31, or 34 to 12),

$$= -\, (g_{3\gamma} g_{4\delta} B^{\gamma\delta})/\sqrt{(-g)} \quad \text{(12 terms)},$$

i.e.,

$$B_{34} = -\, g_{3\gamma} g_{4\delta} B^{\gamma\delta}.$$

Other components can be treated likewise and the theorem proved.

We have so far dealt with these tensors whose transformation equations involve coefficients which are not constants but

functions of the co-ordinates, so as to parallel and extend certain results in Chapter VI., but we have not yet touched on differentiation. We saw in that chapter that simple differentiation yields further tensors directly. For the more general tensors now introduced this is not true, except in one case, viz. the gradient of a scalar function. But nevertheless a process which depends upon differentiation can be applied to these tensors in order to yield further tensors. This process, called "covariant derivation," is of the utmost importance in the application of tensor theory to General Relativity, where we have to express laws of nature in the form of covariant differential equations.

Note on the Terms "Covariant" and "Contravariant."

Let us suppose that at a point in space-time we have chosen four fundamental *unit vectors* which we shall denote by $\mathbf{U}_1$, $\mathbf{U}_2$, $\mathbf{U}_3$, $\mathbf{U}_4$. The vector displacement, whose components are δx_1, δx_2, δx_3, δx_4, is equal to the vectorial sum

$$\delta x_1 \mathbf{U}_1 + \delta x_2 \mathbf{U}_2 + \delta x_3 \mathbf{U}_3 + \delta x_4 \mathbf{U}_4 \quad . \quad . \quad . \quad (1)$$

If we transform to another set of fundamental unit vectors $\mathbf{U}_1'$, $\mathbf{U}_2'$, $\mathbf{U}_3'$, $\mathbf{U}_4'$, and the components of the same vector displacement are $\delta x_1'$, $\delta x_2'$, $\delta x_3'$, $\delta x_4'$, then (1) is equal to

$$\delta x_1' \mathbf{U}_1' + \text{etc.} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If now the equations (12) of the chapter hold between the components δx_λ and $\delta x_\lambda'$, it is easy to prove that the relations between the two sets of fundamental vectors are

$$\left.\begin{aligned} \mathbf{U}_\lambda &= a_{\alpha\lambda} \mathbf{U}_\alpha' \\ \text{or} \qquad \mathbf{U}_\lambda' &= b_{\lambda\alpha} \mathbf{U}_\alpha' \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

It is clear, therefore, that the components of the vectors, which are affected by a suffix in the text, are variables which are transformed cogrediently to the fundamental unit-vectors, and so vectors possessing such components are called covariant vectors. On the other hand, the components of the vectors which are affected by the affix are transformed contragrediently to the fundamental unit-vectors (this is so because of the identity of the vector $A^\alpha \mathbf{U}_\alpha$ with $A'^\alpha \mathbf{U}_\alpha'$), and vectors with such components are termed *contravariant* vectors.

CHAPTER IX.

GEODESICS IN SPACE-TIME.

LET P and Q be two point-instants in space-time, and consider any world-line joining them. Near to this world-line let another world-line be drawn, also joining P and Q. To any point-instant A on the first world-line let there correspond a point-instant A′ on the second world-line, such that the separation between P and A along the first line is equal to the separation between P and A′ along the second, i.e.,

$$\int_{PA} ds = \int_{PA'} ds.$$

In general, the point on the second line corresponding in this way to Q, regarded as on the first, is not Q itself; let it be Q′.

Writing $2L$ for the quadratic function in $dx_1/ds, \ldots, dx_4/ds$, with which we are now familiar, viz., $g_{\alpha\beta}\dot{x}_\alpha\dot{x}_\beta$ (the dot signifying the total differential coefficient dx_λ/ds), and which is everywhere equal to unity, we have

$$\partial L/\partial x_a \,.\, \delta x_a + \partial L/\partial \dot{x}_a \,.\, \delta \dot{x}_a = 0$$

where δx_λ refers to the first order change in the co-ordinates on moving from a point A on one world-line to the corresponding point A′ on the second; and $\delta\dot{x}_\lambda$ refers to the changes in dx_λ/ds (which are analogous to the direction cosines at a point in a curve in two or three dimensions). We can analyse $\delta\dot{x}_\lambda$ more closely by considering a point B on the first line adjacent to A, and B′, its corresponding point on the second line, so that the co-ordinates of B are $x_\lambda + dx_\lambda/ds \,.\, ds$, etc., and of B′ are $x_\lambda + \delta x_\lambda + d(x_\lambda + \delta x_\lambda)/ds \,.\, ds$, etc., the differential ds being the separation between A and B or A′ and B′, for by the method of correspondence used these separations are equal. But the co-ordinates of B′ can also be derived by varying those of B; so they are

$$(x_\lambda + dx_\lambda/ds \,.\, ds) + \delta(x_\lambda + dx_\lambda/ds \,.\, ds).$$

Hence, by comparison of the two results, we see that

$$d(\delta x_\lambda)/ds = \delta(dx_\lambda/ds) \\ = \delta\dot{x}_\lambda.$$

Hence

$$\partial L/\partial x_a \,.\, \delta x_a + \partial L/\partial \dot{x}_a \,.\, d(\delta x_a)/ds = 0.$$

The second term on the left-hand side is equal to

$$d(\partial L/\partial \dot{x}_a \,.\, \delta x_a)/ds - \delta x_a d(\partial L/\partial \dot{x}_a)/ds.$$

Therefore

$$(\partial L/\partial x_a - d(\partial L/\partial \dot{x}_a)/ds)\delta x_a + d(\partial L/\partial \dot{x}_a \,.\, \delta x_a)/ds = 0.$$

If this expression is multiplied by ds and integrated along the world-line PABQ from P to Q, we obtain

$$\int_{PQ}(\partial L/\partial x_a - d(\partial L/\partial \dot{x}_a)/ds)\delta x_a ds = - \partial L/\partial \dot{x}_a \,.\, \delta x_a.$$

Where, *on the right-hand side*, $\partial L/\partial \dot{x}_a$ is estimated at Q and δx_a refers to the differences between the co-ordinates of Q and those of Q′.

Now if the original world-line PAQ happens to be a "geodesic" (i.e., analogous to a shortest or longest length across a surface in two dimensions between two points), so that $\int_{PQ} ds$ is maximum, or minimum, or "stationary" along this line, then $\int_{PQ} ds$ along *any arbitrary* adjacent line differs from the integral along the first by a quantity of the second order. Consequently, to this order Q coincides with Q′; or, on the right-hand side above, δx_a are all of the second order, and so this side can be equated to zero.

Thus the integral on the left-hand side is zero when integrated along a geodesic. But the δx_λ in each element of the integrand is entirely arbitrary as the total separation along the geodesic is stationary for variation to *any arbitrary* neighbouring line. It follows that the factor multiplied by each of the δx_λ is also zero. So finally we arrive at the result that

$$\partial L/\partial x_\lambda - d(\partial L/\partial \dot{x}_\lambda)/ds = 0,$$

provided d/ds refers to differentiation along a geodesic line. In other words, these four equations are the differential equations of a geodesic line in the continuum.

Since

$$\partial L/\partial x_\lambda = \dot{x}_\alpha \dot{x}_\beta \partial g_{\alpha\beta}/\partial x_\lambda \qquad (16 \text{ terms})$$
$$\partial L/\partial \dot{x}_\lambda = g_{\lambda\alpha}\dot{x}_\alpha \qquad (4 \text{ terms})$$

and $\quad d(\partial L/\partial \dot{x}_\lambda)/ds = g_{\lambda\alpha}x_\alpha + \dot{x}_\alpha \dot{x}_\beta \partial g_{\lambda\alpha}/\partial x_\beta \qquad (20 \text{ terms}),$

it is not difficult to collect terms and show that these four differential equations are

$$g_{\lambda\alpha}\ddot{x}_\alpha + [\alpha\beta, \lambda]\dot{x}_\alpha \dot{x}_\beta = 0 \quad (4 \text{ terms in } \ddot{x}, 16 \text{ in } \dot{x}\dot{x}),$$

where

$$[\mu\nu\ \lambda] = \tfrac{1}{2}(\partial g_{\mu\lambda}/\partial x_\nu + \partial g_{\nu\lambda}/\partial x_\mu - \partial g_{\mu\nu}/\partial x_\lambda).$$

$[\mu\nu, \lambda]$ is called Christoffel's three-index symbol of the first order.

These four equations (obtained by putting $\lambda = 1, 2, 3, 4$ in succession) may be looked upon as four simultaneous equations, $x_1, \ddot{x}_2, \ddot{x}_3, \dot{x}_4$. To solve them for $\ddot{x}_1$, say, we multiply them by $g^{11}, g^{21}, g^{31}, g^{41}$ respectively and add.

Since $g^{\alpha 1}g_{\alpha 1} = 1$ and $g^{\alpha 1}g_{\alpha 2} = 0$, etc., we obtain

$$\ddot{x}_1 + (g^{11}[\alpha\beta, 1] + g^{21}[\alpha\beta, 2] + g^{31}[\alpha\beta, 3] + g^{41}[\alpha\beta, 4])\dot{x}_\alpha \dot{x}_\beta = 0$$

or

$$\ddot{x}_1 + \{\alpha\beta, 1\}\dot{x}_\alpha \dot{x}_\beta = 0,$$

where

$$\{\mu\nu, \lambda\} = g^{\lambda\alpha}[\mu\nu, \alpha].$$

$\{\mu\nu, \lambda\}$ is called Christoffel's three-index symbol of the second order.

This expresses $\ddot{x}_1$ in terms of a quadratic function of the $\dot{x}_\lambda$, and we have the differential equations of a geodesic in their most compact form as

$$\ddot{x}_\lambda + \{\alpha\beta, \lambda\}\dot{x}_\alpha \dot{x}_\beta = 0, \qquad (1)$$

where λ is put equal to 1, 2, 3, 4 in succession.

Covariant Derivatives of Tensors.

It was stated in the last chapter that if A_λ is a tensor of the first order the sixteen partial differential coefficients $\partial A_\lambda/\partial x_\mu$ do not constitute a tensor of the second order, as is the case when dealing with the general Lorentz transformation. As a matter of fact, it appears that the sixteen expressions

$$\partial A_\lambda/\partial x_\mu - \{\lambda\mu, \alpha\}A_\alpha$$

are components of a covariant tensor of the second order. Each of these components consist of a four term expression linear in A_1, A_2, A_3, A_4, in addition to the differential coefficient.

Similarly if A^λ is a contravariant tensor of the first order, it appears that the expressions

$$\partial A^\lambda/\partial x_\mu + \{\alpha\mu, \lambda\}A^\alpha$$

constitute a *mixed* tensor of the second order. These derivatives are called "covariant" derivatives because in each case the number of *suffixes* attached to symbol representing the derived tensor is greater by unity than the corresponding number for the original tensor.

We can prove these statements by beginning with a special type of vector and afterwards generalising in an obvious way. Thus, taking the gradient of a scalar function $\phi(x_1, x_2, x_3, x_4)$, we know that $\partial\phi/\partial x_\lambda$ is a covariant vector, and

$$\partial\phi/\partial x_\alpha \,.\, dx_\alpha/ds$$

is invariant, where dx_λ/ds are the "direction cosines" of a geodesic at the point (x_1, x_2, x_3, x_4) and constitute a contravariant vector. In fact, geodesic lines remain geodesic in any space-time frame of axes. This property which a world-line possesses in virtue of the separation along it being stationary, is independent of axes, and so dx_λ/ds are bound to transform into dx_λ'/ds along a geodesic after any transformation. Differentiating again with respect to s totally, we see that

$$d(\partial\phi/\partial x_\alpha \,.\, dx_\alpha/ds)/ds$$

is also invariant, since it is the limit of the invariant value of the quantity $\partial\phi/\partial x_\alpha \,.\, dx_\alpha/ds$, at a point-instant B minus the invariant value of the same quantity at a neighbouring point-instant A on a geodesic, divided by the invariant value of ds along AB. Hence

$$\partial\phi/\partial x_\alpha \,.\, d^2x_\alpha/ds^2 + \partial^2\phi/\partial x_\alpha\partial x_\beta \,.\, dx_\alpha/ds \,.\, dx_\beta/ds$$

is invariant.

But since $\ddot{x}_\lambda$ and $\dot{x}_\lambda$ refer to total differentiation along a geodesic line, therefore $\ddot{x}_\lambda = -\{\alpha\beta, \lambda\}\dot{x}_\alpha\dot{x}_\beta$, and so

$$(\partial^2\phi/\partial x_\alpha\partial x_\beta - \{\alpha\beta, \gamma\}\partial\phi/\partial x_\gamma)\dot{x}_\alpha\dot{x}_\beta$$

is invariant.

Now $\dot{x}_\lambda$ is a contravariant vector, and so the outer product $\dot{x}_\lambda\dot{x}_\mu$ constitutes a contravariant tensor of the second order. Further, since there are an infinite number of geodesics through a given point-instant, $\dot{x}_\lambda\dot{x}_\mu$ may be regarded as *any arbitrary* tensor in the above. In consequence of this invariance and one of the theorems in Chapter VIII., it follows that

$$\partial^2\phi/\partial x_\lambda \partial x_\mu - \{\lambda\mu, \alpha\}\partial\phi/\partial x_\alpha$$

constitute a covariant tensor of the second order, which is the covariant derivative of the covariant vector $\partial\phi/\partial x_\lambda$.

Let A_λ be any covariant vector, and let ϕ_1, ϕ_2, ϕ_3, ϕ_4 be any four *scalar* functions of x_1, x_2, x_3, x_4. After transformation let

$$A_1(x_1, x_2, x_3, x_4) = B_1(x_1', x_2', x_3', x_4'), \text{ etc.,}$$

and $$\phi(x_1, x_2, x_3, x_4) = \psi(x_1', x_2', x_3', x_4'), \text{ etc.}$$

It is very necessary to note that B_1 is *not* A_1', etc., for, of course, $A_1' = b_{1\alpha}A_\alpha$ and $\neq B_1$; in fact, B_1, B_2, B_3, B_4 are not the components in accented co-ordinates of the vector A_λ. Consider the vector

$$A_1\partial\phi_1/\partial x_\lambda + A_2\partial\phi_2/\partial x_\lambda + A_3\partial\phi_3/\partial x_\lambda + A_4\partial\phi_4/\partial x_\lambda,$$

which transforms into the vector

$$B_1\partial\psi_1/\partial x_\lambda' + \text{etc.,}$$

i.e., $$B_\beta\partial\psi_\beta/\partial x_\lambda' = b_{\lambda\alpha}A_\beta\partial\phi_\beta/\partial x_\alpha.$$

Now each term of $A_\beta\partial\phi_\beta/\partial x_\lambda$ is a definite multiple of a gradient, and therefore has a covariant derivative; therefore the vector $A_\beta\partial\phi_\beta/\partial x_\lambda$ has a covariant derivative. By putting $\phi_1 = x_1$, $\phi_2 = x_2$, $\phi_3 = x_3$, $\phi_4 = x_4$ we see that the vector $A_\beta\partial\phi_\beta/\partial x_\lambda$ becomes the vector A_λ. Hence A_λ has a derivative which is a covariant tensor of the second order, viz.,

$$\partial A_\lambda/\partial x_\mu - \{\lambda\mu, \alpha\}A_\alpha.$$

We shall denote it by the symbol $(A_\lambda)_\mu$. Incidentally, we have also proved that any covariant vector can be expressed as the sum of four special vectors of the gradient type.

To deal with covariant tensors of the second order we begin with an outer product of two covariant vectors A_λ and B_λ. The outer product of the $(A_\lambda)_\mu$ by B_λ, and of $(B_\lambda)_\mu$ by A_λ, are individually covariant tensors of the third order. It is also clear that the summation of corresponding components of two tensors *of the same kind and order* will yield a tensor of that kind and order. Hence

$$A_\lambda(B_\mu)_\nu + B_\mu(A_\lambda)_\nu$$

is a covariant tensor of the third order. But this is easily seen to be

$$\partial(A_\lambda B_\mu)/\partial x_\nu - \{\lambda\nu, \alpha\}A_\alpha B_\mu - \{\mu\nu, \alpha\}A_\lambda B_\alpha.$$

Now in a manner similar to that adopted above we can show that any covariant tensor of the second order is the sum of sixteen special tensors which are each outer products of covariant vectors. In fact, if $A_{\lambda\mu}$ is the tensor, it can be expressed as

$$A_{11}\partial\phi_1/\partial x_\lambda \,.\, \partial\phi_1/\partial x_\mu + A_{12}\partial\phi_1/\partial x_\lambda \,.\, \partial\phi_2/\partial x_\mu + \text{etc.}$$

where $\qquad \phi_1 = x_1,\ \phi_2 = x_2,\ \phi_3 = x_3,\ \phi_4 = x_4.$

So the process of covariant derivation can be extended to any covariant tensor of the second order and yields

$$\partial A_{\lambda\mu}/\partial x_\nu - \{\lambda\nu,\ \alpha\}A_{\alpha\mu} - \{\mu\nu,\ \alpha\}A_{\lambda\alpha},$$

which we denote by $(A_{\lambda\mu})_\nu$.

(The position of the dummy suffix must be carefully noted in each of the last two terms, for in general $A_{\lambda\mu} \neq A_{\mu\lambda}$.)

These results can easily be extended to higher orders; for instance, the fourth order covariant derivative of $A_{\lambda\mu\nu}$, i.e., $(A_{\lambda\mu\nu})_\kappa$, is

$$\partial A_{\lambda\mu\nu}/\partial x_\kappa - \{\lambda\kappa,\ \alpha\}A_{\alpha\mu\nu} - \{\mu\kappa,\ \alpha\}A_{\lambda\alpha\nu} - \{\nu\kappa,\ \alpha\}A_{\lambda\mu\alpha}.$$

If we wish to deal with contravariant or mixed tensors we can do so most readily by the method of "association" referred to at the end of the last chapter. An alternative way of expressing the relation between the two Christoffel symbols is required for our purpose. Thus since

$$\{\lambda\mu,\ \nu\} = g^{\nu\alpha}\,[\lambda\mu,\ \alpha]$$

it follows that
$$g_{\kappa\beta}\{\lambda\mu,\ \beta\} = g_{\kappa\beta}g^{\beta\alpha}[\lambda\mu,\ \alpha] = [\lambda\mu,\ \kappa]$$

for
$$g_{\kappa\beta}g^{\alpha\beta} = 1 \text{ if } \alpha = \kappa$$
$$= 0 \text{ if } \alpha \neq \kappa.$$

So that
$$[\lambda\mu,\ \nu] = g_{\nu\alpha}\{\lambda\mu,\ \alpha\}.$$

To proceed, let A^λ be a contravariant vector; then $g_{\lambda\alpha}A^\alpha$ is the "associated" covariant vector. Hence

$$\partial(g_{\kappa\alpha}A^\alpha)/\partial x_\mu - \{\kappa\mu,\ \beta\}g_{\beta\alpha}A^\alpha$$

is the $\kappa\mu$ component of a covariant tensor of the second order, i.e.,

$$g_{\kappa\alpha}\partial A^\alpha/\partial x_\mu + (\partial g_{\kappa\alpha}/\partial x_\mu - [\kappa\mu,\ \alpha])A^\alpha$$

is covariant.

It is easily seen that the factor within the brackets ( ) is the three-index symbol $[\alpha\mu,\ \kappa]$, so

$$g_{\kappa\alpha}\partial A^{\alpha}/\partial x_{\mu} + [\alpha\mu, \kappa]A^{\alpha}$$

is a covariant tensor of the second order. If multiplied (externally) by $g^{\nu\lambda}$, it becomes a mixed tensor of the fourth order. On contracting by putting $\nu = \kappa = \beta$, we obtain a mixed tensor of the second order. But $g^{\beta\lambda}[\alpha\mu, \beta] = \{\alpha\mu, \lambda\}$, and $g^{\beta\lambda}g_{\beta\alpha} = 1$ if $\alpha = \lambda$, and $= 0$ if $\alpha \neq \lambda$. Hence

$$\partial A^{\lambda}/\partial x_{\mu} + \{\alpha\mu, \lambda\}A^{\alpha}$$

is a mixed tensor, and is the covariant derivative of A^{λ}, denoted by $(A^{\lambda})_{\mu}$.

Similarly the covariant derivative of $A^{\lambda\mu}$, i.e: $(A^{\lambda\mu})_{\nu}$, is

$$\partial A^{\lambda\mu}/\partial x_{\nu} + \{\alpha\nu, \lambda\}A^{\alpha\mu} + \{\alpha\nu, \mu\}A^{\lambda\alpha}.$$

The covariant derivative of the mixed tensor $A_{\lambda}{}^{\mu}$, i.e. $(A_{\lambda}{}^{\mu})_{\nu}$, is

$$\partial A_{\lambda}{}^{\mu}/\partial x_{\nu} - \{\lambda\nu, \alpha\}A_{\alpha}{}^{\mu} + \{\alpha\nu, \mu\}A_{\lambda}{}^{\alpha}.$$

Differentials of the Fundamental Tensors.

Since $g_{\mu\beta}g^{\nu\beta}$ is constant (being unity if $\mu = \nu$, and zero if $\mu \neq \nu$), it follows that on varying the values of the co-ordinates in the $g_{\lambda\mu}$-functions,

$$g^{\nu\beta}\delta g_{\mu\beta} = - g_{\mu\beta}\delta g^{\nu\beta}.$$

Multiply each side by $g_{\lambda\nu}$ and sum for the four values of ν; we obtain

$$g_{\lambda\alpha}g^{\alpha\beta}\delta g_{\mu\beta} = - g_{\lambda\alpha}g_{\mu\beta}\delta g^{\alpha\beta}$$

i.e.,

$$\delta g_{\lambda\mu} = - g_{\lambda\alpha}g_{\mu\beta}\delta g^{\alpha\beta};$$

or, in terms of partial differential coefficients of the $g_{\lambda\mu}$-functions, we have

$$\partial g_{\lambda\mu}/\partial x_{\nu} = - g_{\lambda\alpha}g_{\mu\beta}\partial g^{\alpha\beta}/\partial x_{\nu}, \quad . \quad . \quad . \quad (2)$$

which expresses any partial differential coefficient of a component of the covariant fundamental tensor in terms of the ten partial differential coefficients of the components of the contravariant tensor *with respect to the same co-ordinate.*

In a similar manner we can show that

$$\partial g^{\lambda\mu}/\partial x_{\nu} = - g^{\lambda\alpha}g^{\mu\beta}\partial g_{\alpha\beta}/\partial x_{\nu} \quad . \quad . \quad . \quad (3)$$

It is easy to show that

$$\partial g_{\lambda\mu}/\partial x_{\nu} = [\lambda\nu, \mu] + [\mu\nu, \lambda], \quad . \quad . \quad . \quad (4)$$

and so by (3)

$$\partial g^{\lambda\mu}/\partial x_\nu = - g^{\lambda a}g^{\mu\beta}([a\nu, \beta] + [\beta\nu, a]) \qquad \text{(32 terms)}$$
$$= - g^{\lambda a}\{a\nu, \mu\} - g^{\mu\beta}\{\beta\nu, \lambda\};$$

or, since the particular letter chosen for a dummy index does not matter,

$$\partial g_{\lambda\mu}/\partial x_\nu = - (g^{\lambda a}\{a\nu, \mu\} + g^{\mu a}\{a\nu, \lambda\}) \quad . \quad . \quad (5)$$

If $|k|$ is the determinant of a series of constituents k_{11}, k_{12}, . . ., k_{44} and K_{11}, K_{12}, . . ., K_{44} represent the corresponding co-factors, it is known by the theory of determinants that if the constituents *individually* receive small variations, then the variation of $|k|$ is equal to

$$K_{11}\delta k_1 + K_{12}\delta k_2 + \ldots + K_{44}\delta k_{44}.$$

Hence, since $gg^{\lambda\mu}$ is the co-factor of $g_{\lambda\mu}$ in the determinant g, it follows that

$$\delta g = gg^{a\beta}\delta g_{a\beta}$$

i.e.,
$$\delta \log g = \delta g/g = g^{a\beta}\delta g_{a\beta} \quad . \quad . \quad . \quad (6)$$

or, if we wish to deal with the logarithm of an essentially positive quantity,

$$\delta \log(-g) = g^{a\beta}\delta g_{a\beta},$$

so that
$$\partial \log(-g)/\partial x_\lambda = g^{a\beta}\partial g_{a\beta}/\partial x_\lambda \quad . \quad . \quad (7)$$

Now
$$g^{a\beta}g_{a\beta} = 4,$$

hence
$$g_{a\beta}\partial g^{a\beta}/\partial x_\lambda + g^{a\beta}\partial g_{a\beta}/\partial x_\lambda = 0,$$

and therefore
$$\partial \log(-g)/\partial x_\lambda = - g_{a\beta}\partial g^{a\beta}/\partial x_\lambda \quad . \quad . \quad (7\text{A})$$

Combining (7A) with (5) we obtain

$$\partial \log(-g)/\partial x_\lambda = g_{a\beta}(g^{a\gamma}\{\gamma\lambda, \beta\} + g^{\beta\gamma}\{\gamma\lambda, a\})$$
$$= \{\beta\lambda, \beta\} + \{a\lambda, a\}$$
$$= 2\{\lambda a, a\} \quad . \quad . \quad . \quad . \quad (8)$$

an extremely useful result for later work.

Another way of writing (8) is

$$\partial \log(-g)^{\frac{1}{2}}/\partial x_\lambda = (-g)^{-\frac{1}{2}}\partial(-g)^{\frac{1}{2}}/\partial x_\lambda = \{\lambda a, a\} \quad . \quad (8\text{A})$$

Curl, Divergence, and Lorentzian.

We can now proceed to deal with the analogues of certain vector operations employed in Chapter VI.

In the first place, it is easy to prove that if $A_{\lambda\mu}$ is the $\lambda\mu$-component of any covariant tensor of the second order, and

$A_{\mu\lambda}$ is its $\mu\lambda$-component, then $A_{\lambda\mu}-A_{\mu\lambda}$ constitutes the components of a six-vector when $\lambda\mu$ goes through the sequence 23, 31, 12, 14, 24, 34. In particular, $A_\lambda B_\mu - A_\mu B_\lambda$ is a six-vector, and is analogous to the six-vector product of two four-vectors in Chapter VI.

Let $(A_\lambda)_\mu$ be the covariant derivative of A_λ, then

$$(A_\lambda)_\mu - (A_\mu)_\lambda = \partial A_\lambda/\partial x_\mu - \partial A_\mu/\partial x_\lambda$$

for
$$\{\lambda\mu, \alpha\} = \{\mu\lambda, \alpha\}.$$

It follows that $\partial A_\lambda/\partial x_\mu - \partial A_\mu/\partial x_\lambda$ is the component of a six-vector which is the " Curl " or " Rotation " of A_λ.

It is perhaps as well to point out that the operation " Curl," when applied to a *contravariant* vector, does not yield a six-vector, because $\{\alpha\mu, \lambda\} \neq \{\alpha\lambda, \mu\}$.

We can, however, obtain a scalar Divergence of a contravariant vector, for $(A^\lambda)_\mu$ is a mixed tensor, and so by contraction $(A^\alpha)_\alpha$ or

$$\partial A^\alpha/\partial x_\alpha + \{\beta\alpha, \alpha\}A^\beta$$

is invariant.

Hence by (8A), writing q for $(-g)^{\frac{1}{2}}$,

$$\partial A^\beta/\partial x_\beta + q^{-1}A^\alpha\partial q/\partial x$$

is invariant ; or, since the letter used for a dummy index does not matter,

$$q^{-1}\partial(qA^\alpha)/\partial x_\alpha \quad . \quad . \quad . \quad . \quad (9)$$

is invariant. This is the analogue of Div **A** of Chapter VI.

There is no corresponding function for a covariant vector, unless we introduce it *via* the associated contravariant vector $g^{\lambda\alpha}A_\alpha$.

We now come to analogues of the four-vector Lorentzian of a six-vector.

Let $A_{\lambda\mu}$ be an *antisymmetric* covariant tensor or covariant six-vector, then $(A_{\lambda\mu})_\nu$ is a covariant tensor of the third order (the components for which $\lambda = \mu$ are zero). A study of the transformation equations shows that

$$(A_{\lambda\mu})_\nu + (A_{\mu\nu})_\lambda + (A_{\nu\lambda})_\mu$$

is an antisymmetric tensor of the third order.

Since $\{\lambda\mu, \alpha\} = \{\mu\lambda, \alpha\}$ and $A_{\lambda\mu} = - A_{\mu\lambda}$ it easily follows that

$$\partial A_{\lambda\mu}/\partial x_\nu + \partial A_{\mu\nu}/\partial x_\lambda + \partial A_{\nu\lambda}/\partial x_\mu$$

is an antisymmetric tensor of the third order if $A_{\lambda\mu}$ is antisymmetric.

From a result in Chapter VIII. (p. 186), we see that if $A_{\lambda\mu}$ is a covariant six-vector, there exists a contravariant vector B^κ, such that

$$\begin{aligned} qB^\kappa &= (A_{\lambda\mu})_\nu + (A_{\mu\nu})_\lambda + (A_{\nu\lambda})_\mu \\ &= \partial A_{\lambda\mu}/\partial x_\nu + \partial A_{\mu\nu}/\partial x_\lambda + \partial A_{\nu\lambda}/\partial x_\mu \quad . \quad . \quad (10) \end{aligned}$$

where κ, λ, μ, ν are replaced in turn by 1, 2, 3, 4 ; 2, 3, 1, 4 ; 3, 1, 2, 4 ; 4, 3, 2, 1.

Also, in Chapter VIII. it has been proved that there exists a contravariant six-vector $R^{\lambda\mu}$ reciprocal to $A_{\lambda\mu}$ such that

$$A_{23} = qR^{14}$$

and five similar equations.

Introducing these into (10), we easily prove that

$$q^{-1}\partial(qR^{\lambda\alpha})/\partial x_\alpha$$

is a contravariant vector which we can denote by the name "Lorentzian of $R^{\lambda\mu}$."

This last result can, as a matter of fact, be proved directly without any appeal to reciprocity.

Thus let $A^{\lambda\mu}$ be a contravariant six-vector, then

$$(A^{\lambda\mu})_\nu = \partial A^{\lambda\mu}/\partial x_\nu + \{\alpha\nu, \lambda\}A^{\alpha\mu} + \{\alpha\nu, \mu\}A^{\lambda\alpha}$$

is a mixed tensor of the third order. By contraction we obtain a contravariant vector $(A^{\lambda\beta})_\beta$ which is equal to

$$\partial A^{\lambda\beta}/\partial x_\beta + \{\alpha\beta, \lambda\}A^{\alpha\beta} + \{\alpha\beta, \beta\}A^{\lambda\alpha}.$$

Owing to the antisymmetry of $A^{\lambda\mu}$ and the equality of $\{\mu\nu, \lambda\}$ and $\{\nu\mu, \lambda\}$, the second term, when summed for the α and β affixes, vanishes. Hence

$$\begin{aligned} (A^{\lambda\beta})_\beta &= \partial A^{\lambda\beta}/\partial x_\beta + \{\alpha\beta, \beta\}A^{\lambda\alpha} \\ &= \partial A^{\lambda\beta}/\partial x_\beta + q^{-1}A^{\lambda\alpha}\partial q/\partial x_\alpha. \end{aligned}$$

Or, altering the dummy index, we see that

$$q^{-1}\partial(qA^{\lambda\alpha})/\partial x_\alpha \quad . \quad . \quad . \quad . \quad . \quad (11)$$

is a contravariant vector, being, in fact, the contracted covariant derivative $(A^{\lambda\alpha})_\alpha$ of the six-vector $A^{\lambda\mu}$.

There still remains an analogous operation for a mixed tensor of the second order, and in this case it is not of necessity antisymmetric. Thus

$$(A_\lambda{}^\mu)_\nu = \partial A_\lambda{}^\mu/\partial x_\nu + \{\alpha\nu, \mu\}A_\lambda{}^\alpha - \{\lambda\nu, \alpha\}A_\alpha{}^\mu$$

is a mixed tensor of the third order.

By contraction we obtain a covariant vector $(A_\lambda{}^\beta)_\beta$ which is

$$\partial A_\lambda{}^\beta/\partial x_\beta + \{\alpha\beta, \beta\}A_\lambda{}^\alpha - \{\lambda\beta, \alpha\}A_\alpha{}^\beta$$

or

$$\partial A_\lambda{}^\beta/\partial x_\beta + q^{-1}A_\lambda{}^\alpha\partial q/\partial x_\alpha - \{\lambda\beta, \alpha\}A_\alpha{}^\beta.$$

After an obvious adaptation of the dummy indexes we have as the contracted covariant derivative of $A_\lambda{}^\mu$ the vector

$$(A_\lambda{}^\alpha)_\alpha = q^{-1}\partial(qA_\lambda{}^\alpha)\partial x_\alpha - \{\lambda\alpha, \beta\}A_\beta{}^\alpha \quad . \quad . \quad (12)$$

We can express the last term in this result somewhat differently by employing the associated covariant or contravariant tensor; for

$$\begin{aligned}\{\lambda\alpha, \beta\}A_\beta{}^\alpha &= g^{\beta\gamma}[\lambda\alpha, \gamma]A_\beta{}^\alpha \\ &= [\lambda\alpha, \gamma]A^{\alpha\gamma}.\end{aligned}$$

Hence

$$(A_\lambda{}^\alpha)_\alpha = q^{-1}\partial(qA_\lambda{}^\alpha)/\partial x_\alpha - [\lambda\alpha, \beta]A^{\alpha\beta} \quad . \quad . \quad (12\text{A})$$

Also,

$$[\lambda\alpha, \beta]A^{\alpha\beta} = g^{\alpha\gamma}g^{\rho\delta}\, A_{\gamma\delta}[\lambda\alpha, \beta]$$

$$\begin{aligned}&= \tfrac{1}{2}A_{\gamma\delta}g^{\gamma\alpha}g^{\beta\delta}(\partial g_{\lambda\beta}/\partial x_\alpha - \partial g_{\lambda\alpha}/\partial x_\beta + \partial g_{\alpha\beta}/\partial x_\lambda) \\ &= \tfrac{1}{2}A_{\gamma\delta}g^{\gamma\alpha}g^{\beta\delta}\partial g_{\alpha\beta}/\partial x_\lambda \text{ (since the other terms cancel in the summation)} \\ &= -\tfrac{1}{2}A_{\gamma\delta}\partial g^{\gamma\delta}/\partial x_\lambda.\end{aligned}$$

Hence

$$(A_\lambda{}^\alpha)_\alpha = q^{-1}\partial(qA_\lambda{}^\alpha)/\partial x_\alpha + \tfrac{1}{2}A_{\alpha\beta}\partial g^{\alpha\beta}/\partial x_\lambda \quad . \quad . \quad (12\text{B})$$

Collecting these latter results, we see that there exist functions which we can call:

(1) The six-vector Curl or Rotation of a covariant vector.

(2) The invariant scalar Divergence of a contravariant vector.

(3) The contravariant vector which is the Lorentzian of a contravariant six-vector, or of the reciprocal of a covariant six-vector.

(4) The covariant vector which is the Lorentzian of a mixed tensor of the second order.

The "Riemann-Christoffel" Tensor.

Having thus established the method for the covariant derivation of tensors from tensors, it seems very natural to apply the process itself to the fundamental tensor $g_{\lambda\mu}$. The result, however, is disappointing, for little trouble is required to show that $(g_{\lambda\mu})_\nu$ is zero.

Yet it is of vital importance for the development of the analysis of a four-dimensional continuum to obtain a tensor derived by differentiation from the fundamental tensor. The following method arrives at such a tensor called after the two men who independently discovered it.

Suppose we apply the method of covariant derivation twice to *any arbitrary* covariant vector. The first derivation gives

$$(A_\lambda)_\mu = \partial A_\lambda/\partial x_\mu - \{\lambda\mu, \beta\}A_\beta,$$

and the second derivation gives

$$\begin{aligned}((A_\lambda)_\mu)_\nu &= \partial(A_\lambda)_\mu/\partial x_\nu - \{\lambda\nu, \alpha\}(A_\alpha)_\mu - \{\mu\nu, \alpha\}(A_\lambda)_\alpha \\ &= \partial(\partial A_\lambda/\partial x_\mu - \{\lambda\mu, \beta\}A_\beta)/\partial x_\nu \\ &\quad - \{\lambda\nu, \alpha\}(\partial A_\alpha/\partial x_\mu - \{\alpha\mu, \beta\}A_\beta) \\ &\quad - \{\mu\nu, \alpha\}(\partial A_\lambda/\partial x_\alpha - \{\lambda\alpha, \beta\}A_\beta) \\ &= \partial^2 A_\lambda/\partial x_\mu \partial x_\nu \\ &\quad - \{\lambda\mu, \beta\}\partial A_\beta/\partial x_\nu - \{\lambda\nu, \alpha\}\partial A_\alpha/\partial x_\mu - \{\mu\nu, \alpha\}\partial A_\lambda/\partial x_\alpha \\ &\quad + A_\beta\{\lambda\nu, \alpha\}\{\alpha\mu, \beta\} + A_\beta\{\mu\nu, \alpha\}\{\lambda\alpha, \beta\} \\ &\quad - A_\beta \partial\{\lambda\mu, \beta\}/\partial x_\nu.\end{aligned}$$

If we interchange the order of derivation we can write down in a similar way the expression for $((A_\lambda)_\nu)_\mu$, and it is easily seen that the first four terms and the sixth in $((A_\lambda)_\mu)_\nu$ make their appearance in $((A_\lambda)_\nu)_\mu$. As the difference of these derivatives is a tensor, it appears that

$$(\partial\{\lambda\nu, \beta\}/\partial x_\mu - \partial\{\lambda\mu, \beta\}/\partial x_\nu + \{\lambda\nu, \alpha\}\{\alpha\mu, \beta\} - \{\lambda\mu, \alpha\}\{\alpha\nu, \beta\})A_\beta$$

is a covariant tensor of the third order. It is obviously a contracted product of the covariant vector A_κ and a mixed tensor of the fourth order, viz.,

$$\partial\{\lambda\nu, \kappa\}/\partial x_\mu - \partial\{\lambda\mu, \kappa\}/\partial x_\nu + \{\lambda\nu, \alpha\}\{\alpha\mu, \kappa\} - \{\lambda\mu, \alpha\}\{\alpha\nu, \kappa\} \quad (13)$$

which we denote by $R_{\lambda\mu\nu}{}^\kappa$ since it is contravariant as regards the symbol κ. It is obviously a function of the $g_{\lambda\mu}$ functions and of their first and second differential coefficients with respect to the co-ordinates.

We can also derive a covariant tensor $R_{\lambda\mu\nu\kappa}$ of the fourth order by association. It is $g_{\kappa\beta}R_{\lambda\mu\nu}{}^\beta$, and is equal to

$$\begin{aligned}&g_{\kappa\beta}(\partial\{\lambda\nu, \beta\}/\partial x_\mu - \partial\{\lambda\mu, \beta\}/\partial x_\nu + \{\lambda\nu, \alpha\}\{\alpha\mu, \beta\} - \{\lambda\mu, \alpha\}\{\alpha\nu, \beta\}) \\ &= \partial(g_{\kappa\beta}\{\lambda\nu, \beta\})/\partial x_\mu - \{\lambda\nu, \beta\}\partial g_{\kappa\beta}/\partial x_\mu \\ &\quad - \partial(g_{\kappa\beta}\{\lambda\mu, \beta\})/\partial x_\nu + \{\lambda\mu, \beta\}\partial g_{\kappa\beta}/\partial x_\nu \\ &\quad + \{\lambda\nu, \alpha\}[\alpha\mu, \kappa] - \{\lambda\mu, \alpha\}[\alpha\nu, \kappa] \\ &= \partial[\lambda\nu, \kappa]/\partial x_\mu - \partial[\lambda\mu, \kappa]/\partial x_\nu \\ &\quad + \{\lambda\nu, \alpha\}([\alpha\mu, \kappa] - \partial g_{\kappa\alpha}/\partial x_\mu) - \{\lambda\mu, \alpha\}([\alpha\nu, \kappa] - \partial g_{\kappa\alpha}/\partial x_\nu) \\ &= \partial[\lambda\nu, \kappa]/\partial x_\mu - \partial[\lambda\mu, \kappa]/\partial x_\nu + \{\lambda\mu, \alpha\}[\kappa\nu, \alpha] - \{\lambda\nu, \alpha\}[\kappa\mu, \alpha] \quad (14)\end{aligned}$$

The covariant tensor (14) is sometimes denoted by the symbol $(\kappa\lambda\mu\nu)$, called a "four-index" symbol of Christoffel.

A little investigation will show that

$$\begin{aligned}(\kappa\lambda\mu\nu) &= (\lambda\kappa\nu\mu) = (\mu\nu\kappa\lambda) = (\nu\mu\lambda\kappa)\\ &= -(\lambda\kappa\mu\nu) = -(\kappa\lambda\nu\mu) = -(\nu\mu\kappa\lambda) = -(\mu\nu\lambda\kappa),\end{aligned}$$

i.e., interchanging κ and λ or μ and ν alters the sign, and displacing $\kappa\lambda$ to the position of $\mu\nu$ leaves the value unaltered. Further,

$$(\kappa\lambda\mu\nu) + (\kappa\mu\nu\lambda) + (\kappa\nu\lambda\mu) = 0$$

(which gives a relation when three of the index signs are permuted and one is not) is a result which can be easily proved by writing out in full, when it is seen that the terms cancel in pairs.

These equalities show that there are only twenty numerically different components which are not zero. Thus:

Those of	type	$(\kappa\kappa\kappa\kappa)$	are all	identically zero.
,,	,,	$(\kappa\kappa\kappa\lambda)$	,,	,,
,,	,,	$(\kappa\kappa\lambda\lambda)$	,,	,,
,,	,,	$(\kappa\kappa\lambda\mu)$	,,	,,

There remain	6	of the	type	$(\kappa\lambda\lambda\kappa)$	or $R_{\lambda\lambda\kappa\kappa}$
,, ,,	12	,,	,,	$(\kappa\lambda\mu\kappa)$	or $R_{\lambda\mu\kappa\kappa}$
,, ,,	2	,,	,,	$(\kappa\lambda\mu\nu)$	or $R_{\lambda\mu\nu\kappa}$,

i.e., twenty in all which are not zero and are in general numerically different.

Similar conclusions hold for the components of the tensor $R_{\lambda\mu\nu}{}^{\kappa}$.

The Law of Motion of a Particle.

The object of the development of this tensor analysis for a "non-homaloidal" space-time is to obtain a mathematical symbolism suitable for the formulation of laws of nature which will pass the Relativity test in any frame of reference. The test will be satisfied if a law can be expressed as an equality between tensors, or as the equality of some linear function of tensors to zero.

It is very natural to choose (1) as the law of motion of a particle of matter "under no forces," because, in the first place,

$$\ddot{x}_\lambda + \{\alpha\beta, \lambda\}\dot{x}_\alpha\dot{x}_\beta$$

is a contravariant vector,* and, secondly, it is the law of motion in a Galilean frame, for in that case, since $g_{\lambda\mu}$ is zero, the equations reduce to

$$\ddot{x}_\lambda = 0,$$

which we know to be the law of motion from the investigations of the earlier theory of Relativity.

If we choose a frame of reference which is free from gravitation, the path of a particle "under no forces," i.e., unaffected by pressures, pulls, electromagnetic forces, is determined by $\ddot{x}_\lambda = 0$. The path of the same particle, still unaffected by such forces, in another frame is $\ddot{x}_\lambda + \{\alpha\beta, \lambda\}\dot{x}_\alpha\dot{x}_\beta = 0$, where the $g_{\lambda\mu}$ functions which occur in the coefficient $\{\alpha\beta, \lambda\}$ are determined by the law of transformation of co-ordinates from the one frame to the other. This path is, of course, curved and independent of the nature of the particle; it has all the features of a path in a field of gravitation; in this case we call the field "fictitious" or "geometrical" or "non-permanent." Einstein assumes that this is also the equation of motion of a particle "under no forces" in an actual or "permanent" gravitational field. The assumption is based on the Principle of Equivalence, and satisfies the general Principle of Relativity. He can still refer to the particle as "under no forces" (meaning forces arising from material contact or electromagnetic action), because he has by this procedure removed gravitation from the category of force and made its manifestations arise from the metrical relations of space-time, i.e., the expression of the invariant separation between two point-instants in terms of the particular system of co-ordinates chosen. In fact, from the manner in which equation (1) was derived, we can say that the natural path of a particle in a gravitational field is that which makes the separation between two given point-instants of its history the greatest possible. The world-line of the particle is a geodesic. In a region free from permanent gravitation, the natural path between two points is the shortest possible, so that given the time of passage, the separation is a maximum, for

$$\delta s^2 = \delta t^2 - \delta x^2 - \delta y^2 - \delta z^2$$

or

$$\delta s = \delta t(1 - v^2)^{\frac{1}{2}},$$

* $\ddot{x}_\lambda + \{\alpha\beta, \lambda\}\dot{x}_\alpha\dot{x}_\beta = \dot{x}_\alpha(\partial\dot{x}_\lambda/\partial x_\alpha + \{\beta\alpha, \lambda\}x_\beta)$, and the expression in the brackets ( ) is a component of the covariant derivative of the contravariant vector $\dot{x}_\lambda$. Hence the right-hand side is the inner product of a contravariant vector and a mixed tensor of the second order, and so is itself a contravariant vector.

and so, given δt,. δs is a maximum when v is smallest, which is the case for the straight path of shortest length. On transforming co-ordinates, the curved path in the new space frame of reference retains the same character, since δs between two definite events is invariant, and if it is maximum for a definite sequence of events in one frame, it remains so in any other.

It is very necessary to go warily at this point. It is a matter of pure mathematics only to say that (1) is the equation of motion of a particle in a geometrical gravitational field such as we derive by transferring ourselves from a Galilean frame to one in variable motion relative to the former; at least, it is so if we admit that $\ddot{x}_\lambda = 0$ is the law of motion in the Galilean frame. But we invoke the Principle of Equivalence when we say that (1) is the equation of motion in a permanent field of gravitation; for such a field cannot be removed by a transformation of axes in the way in which a geometrical field can be removed.

Einstein's Law of Gravitation.

This brings us at once to the closer consideration of those characteristics which mark off permanent and natural fields of gravitation as a special class from all conceivable fields which might be introduced by all conceivable modes of co-ordinate transformation. Such characteristics must be expressible in the form of equations satisfied by the $g_{\lambda\mu}$ coefficients, equations which indicate the nature of the changes taking place in these coefficients continuously as a point-instant changes in space-time, i.e., differential equations. Such equations abandon all notions of "action at a distance," and determine the motion of a particle at a given position and instant, not by an appeal to a distant body, but to the metrical conditions of the portion of space-time in which this event is located, these conditions being related step by step to conditions round all other point-instants, and ultimately to those regions of space-time which contain the world-lines of all particles of gravitating matter.

We now perceive why it is so important to obtain tensors which involve the differential coefficients of the $g_{\lambda\mu}$-functions. If the law of gravitation is to be subject to the Principle of Relativity it must involve such tensors. We have obtained one such tensor, the "Riemann-Christoffel," and any other tensors which we can derive from it by the methods of contraction and association. Our first impulse is to try the law

$$R_{\lambda\mu\nu}{}^{\kappa} = 0 \quad . \quad . \quad . \quad . \quad (15)$$

which, as has been indicated, is really a collection of twenty independent equations. Unfortunately, this law, although true in any gravitational field which can be derived by a mathematical transformation from a Galilean frame, i.e., in a geometric field, cannot be true in a natural field. To see this, we have only to note that (15) is certainly true in a Galilean frame, because there the $g_{\lambda\mu}$-coefficients are constants ; and since after a transformation the tensor components of $R_{\lambda\mu\nu}{}^{\kappa}$ in the new co-ordinates are linear functions of the components in the old, (15) must still be true in the new co-ordinates. Hence (15) is a *necessary* condition in any frame of reference if the gravitational field in it can be removed *everywhere* by a suitable transformation of co-ordinates. It can also be proved that (15) is a *sufficient* condition for the possibility of such a removal. But it is an outstanding feature of actual gravitational fields that they cannot be removed everywhere by a suitable choice of axes. Fix the axes on a small piece of matter moving naturally through the field, and gravitation is removed in the immediate neighbourhood, but not at places at any moderate distance from the origin. Consequently equation (15) is too stringent a condition to impose upon the $g_{\lambda\mu}$-coefficients. Einstein, realising this, and bearing in mind that any more general equation which would be satisfied both in permanent and non-permanent fields must include (15) as a particular case, has suggested instead of (15) the equations

$$R_{\lambda\mu\alpha}{}^{\alpha} = 0 \quad . \quad . \quad . \quad . \quad (16)$$

i.e., $$R_{\lambda\mu 1}{}^{1} + R_{\lambda\mu 2}{}^{2} + R_{\lambda\mu 3}{}^{3} + R_{\lambda\mu 4}{}^{4} = 0.$$

These are clearly satisfied if the individual $R_{\lambda\mu\nu}{}^{\kappa}$ components are zero, but they are also consistent with other than non-permanent fields. As the process of contraction is involved, the expression equated to zero is a covariant tensor of the second order, and so the equation (16) is covariant and satisfies the Principle of Relativity. This contracted Riemann-Christoffel tensor is usually denoted by the symbol $G_{\lambda\mu}$, and thus Einstein's law of gravitation for points where there is no matter (or electromagnetic energy) is

$$G_{\lambda\mu} = 0 \quad . \quad . \quad . \quad . \quad (16\text{A})$$

Since $G_{\lambda\mu}$ is a symmetric tensor, on account of the symmetry of $g_{\lambda\mu}$, there are ten such equations, but it will appear later that there are four identical relations between the $G_{\lambda\mu}$ components, which therefore reduces the number of independent equations

in (16A) to six. It is clear that the equations involve the differential coefficients of the $g_{\lambda\mu}$-components up to the second order.

Before proceeding further it will be convenient to write them more explicitly.

By (13) (altering the dummy suffix),

$$R_{\lambda\mu\alpha}{}^{\alpha} = \partial\{\lambda\alpha, \alpha\}/\partial x_{\mu} - \partial\{\lambda\mu, \alpha\}/\partial x_{\alpha} + \{\lambda\alpha, \beta\}\{\mu\beta, \alpha\} - \{\lambda\mu, \beta\}\{\beta\alpha, \alpha\}.$$

But by (8),

$$\{\lambda\alpha, \alpha\} = \partial \log q/\partial x_{\lambda}.$$

Therefore

$$G_{\lambda\mu} = -\partial\{\lambda\mu, \alpha\}/\partial x_{\alpha} + \{\lambda\alpha, \beta\}\{\mu\beta, \alpha\} + \partial^2 \log q/\partial x_{\lambda}\partial x_{\mu} - \{\lambda\mu, \alpha\}\partial \log q/\partial x_{\alpha} \quad . \quad (17)$$

We should expect that as Newtonian theory gives such a close approximation between observation and calculation, the equations (16) would degenerate under suitable conditions to the equations of Newtonian theory, and we shall deal with this matter at once. Schwarzschild's method of obtaining an exact solution of Einstein's equations corresponding to the field of a single body such as the sun will be given later; but so as not to interrupt the general development of the theory, we shall, for the time being, make use of the following conclusions, which can be obtained from an inspection of the solution. (See Chapter XIII.) Regarding x_1, x_2, x_3 as Cartesian co-ordinates referred to axes with the Sun as origin and x_4 as time, $g_{\lambda\mu}$ differs from plus or minus unity by quantities of the order M$/r$, if $\lambda = \mu$, where M is the gravitational mass of the sun; the gravitating effect of the planets will not introduce any differences except of a lower order of magnitude. At the orbit of Mercury M$/r$ is of the order 10^{-7}. If $\lambda \neq \mu$ and neither λ nor μ is 4, $g_{\lambda\mu}$ is actually of the order M$/r$. If $\lambda \neq 4$, $g_{\lambda 4}$ may be treated as zero. They are zero in Schwarzschild's and Einstein's solutions, since these solutions refer to a static field. Through the motion of the planets, however, in the frame of reference, the field will alter at each point by relatively small amounts with lapse of time. In any case, $g_{\lambda 4}$ is not only small, but it is also involved in terms of the equations which are of a relatively low order of magnitude, as the reader can verify very easily for himself at a later stage. It has also to be observed that g is equal to -1, or approximately so. As regards the differential coefficients,

$\partial g_{\lambda\mu}/\partial x_\nu$ is of the order M/r^2 if $\lambda, \mu, \nu \neq 4$
$\partial g_{44}/\partial x_\nu$,, ,, ,, ,, ,, $\nu \neq 4$
$\partial g_{\lambda\mu}/\partial x_\nu$ is of a lower order or zero if $\nu = 4$ and λ or $\mu = 1, 2, 3, 4$
,, ,, ,, ,, ,, ,, $\mu = 4$ and λ or $\nu \neq 4$.

From these can be worked out the order of magnitude of the [ ] symbols, and then of the { } symbols.

$\{\lambda\mu, \nu\}$ is of the order M/r^2 if $\lambda, \mu, \nu \neq 4$
$\{44, \nu\}$,, ,, ,, $\nu \neq 4$
$\{\lambda 4, 4\}$,, ,, ,, $\lambda \neq 4$
$\{\lambda 4, \nu\}$ is of a lower order or zero if λ or $\nu \neq 4$
$\{\lambda\mu, 4\}$,, ,, ,, if λ or $\mu \neq 4$
$\{44\ 4\}$,, ,, ,,

In the differential equations of motion for a particle, viz., (1), it will appear in a moment that dx_4/ds is of the order unity in this approximation, and so dx_1/ds, dx_2/ds, dx_3/ds are comparable with the components of a particle's velocity in the field, and so are of the order $(M/r)^{\frac{1}{2}}$ in general.

In the fourth of equations (1) we can neglect those of the terms $\{\alpha\beta, 4\}$ in which both α and β are not 4. Hence

$$\ddot{x}_4 + 2\{\alpha 4, 4\}\dot{x}_\alpha \dot{x}_4 = 0.$$

Now

$$\begin{aligned} \{\alpha 4, 4\} &= g^{4\beta}[\alpha 4, \beta] \\ &= g^{44}[\alpha 4, 4] \\ &= \tfrac{1}{2} g^{44} \partial g_{44}/\partial x_\alpha. \end{aligned}$$

and hence $$\begin{aligned} d^2x_4/ds^2 &= -\, g^{44} dg_{44}/dx_\alpha \, . \, dx_\alpha/ds \, . \, dx_4/ds \\ &= -\, g^{44} \partial g_{44}/ds \, . \, dx_4/ds. \end{aligned}$$

To this order of approximation

$$g^{44} = 1/g_{44}.$$

Hence

$$\frac{d^2x_4/ds^2}{dx_4/ds} = -\frac{dg_{44}/ds}{g_{44}},$$

or $$\log(dx_4/ds) + \log g_{44} = \text{constant},$$

i.e., $$dt/ds \text{ or } dx_4/ds = \text{constant}/g_{44}.$$

This (in general) approximate equation is actually exact in the case of Schwarzschild's solution (see equation (2), Appendix to Chapter XIII.). Since $1 - g_{44}$ approaches zero as we recede from gravitating matter, and as δt approaches δs under similar circumstances, the constant is unity. Hence approximately

$$dt/ds = 1/g_{44}.$$

Near gravitating matter, therefore, $\delta t > \delta s$ (since $g_{44} = 1 - M/r$), or the time of an occurrence in a gravitational field is greater than its proper time, which is the general statement covering the displacement towards the red of the solar lines.

The first three of equations (1) can be thrown into a form involving d/dt and d^2/dt^2 as follows. (We use t as alternative to x_4.) Put $dx_4/ds = \beta$, we obtain

$$\beta^2 dx_\lambda{}^2/dt^2 + \beta d\beta/dt \,.\, dx_\lambda/dt + \{\alpha\beta, \lambda\}\beta^2 dx_\alpha/dt \,.\, dx_\beta/dt = 0.$$

The fourth equation is

$$\beta d\beta/dt + \{\alpha\beta, 4\}\beta^2 dx_\alpha/dt \,.\, dx_\beta/dt = 0.^*$$

Hence

$$d^2x_\lambda/dt^2 = -(\{\alpha\beta, \lambda\} - \{\alpha\beta, 4\})dx_\alpha/dt \,.\, dx_\beta/dt$$

are the three equations of motion ($\lambda = 1, 2, 3$ in succession).

As a first approximation we only keep terms of the order M/r^2 on the right-hand side, and we find

$$d^2x_\lambda/dt^2 = -\{44, \lambda\}.$$

This is so since dx_λ/dt is of the order $(M/r)^{\frac{1}{2}}$ for $\lambda = 1, 2,$ or 3, but of the order unity if $\lambda = 4$.

Now

$$\begin{aligned} \{44, \lambda\} &= g^{\lambda\alpha}[44, \alpha] \\ &= g^{\lambda\lambda}[44, \lambda] \text{ (not summed) approximately} \\ &= -[44, \lambda] \\ &= -\tfrac{1}{2}\partial g_{44}/dx_\lambda. \end{aligned}$$

Hence

$$d^2x_\lambda/dt^2 = \partial\phi/\partial x_\lambda$$

if

$$g_{44} = 1 - 2\phi,$$

which emphasises a result, already known in special cases of "geometric" fields, that g_{44} differs from unity by twice a Newtonian potential function.

Turning now to the first order approximations to Einstein's law of gravitation, let us consider the equation

$$G_{44} = 0\,;$$

or

$$\partial\{44, \alpha\}/\partial x_\alpha - \{4\alpha, \beta\}\{4\beta, \alpha\} = \partial^2 \log q/\partial x_4{}^2 - \{44, \alpha\}\partial \log q/\partial x_\alpha.$$

The first term on the left-hand side involves terms of the order M/r^3 at the highest, i.e., of the order $M/r \,.\, 1/r^2$; the second

* $\beta = dx_4/ds$ is not, of course, to be confused with the β in the index symbols.

term involves terms of the order M^2/r^4 or $(M/r)^2 . 1/r^2$, and so can be neglected in comparison with the first term. On the right-hand side both terms involve no terms of lower magnitude than M^2/r^4. Thus the equation reduces to

$$\partial\{44, \alpha\}/\partial x_\alpha = 0 ;$$

or
$$(\partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + \partial^2/\partial x_3^2)g_{44} = 0,$$

which, in view of the interpretation of g_{44} above, is Laplace's equation for the Newtonian potential function.

It will be as well also to refer to what has been called the " official attitude " of the relativist towards co-ordinates.

We have already seen what the procedure is in a number of cases when dealing with Restricted Relativity, and it is essentially no different in General Relativity. A law of nature is submitted as true; if it be cast in the covariant mould, i.e., if its form be such that it is obviously a set of linear equations between linear tensors, all is well; if this be not so, the equation must be generalised so that it takes on this desirable property and degenerates to the original form in the special frame of reference for which that form is known to be " true to nature." The great strength of the Relativity standpoint is due to its success in divesting all the fundamental laws of Physics of those accidental features which appear in their expressions in terms of some particular type of co-ordinates, chosen in some special frame of reference. By the very nature of tensor analysis in a four-dimensional continuum, the equations are quite symmetrical in the four variables x_1, x_2, x_3, x_4. There is absolutely nothing in the equations themselves to indicate that any particular variable is "time," and that the other three are " space " co-ordinates; or, again, that three of the co-ordinates are Cartesian, polar, cylindrical, ellipsoidal, etc. We have been calling x_4 " time " so far; but that was merely a formal convenience. We could have called x_1 or x_2 or x_3 " time," and chosen x_4 as a " space " co-ordinate. Of course, this arises from the unification of space and time which distinguishes Relativity theory, and is analogous to the same lack of definiteness in the older forms of physical laws as regards Cartesian co-ordinates, where the interchanging of x, y, and z produces no contradiction or error. In fact, x_1, x_2, x_3, x_4 are simply any tetrad of quantities which *uniquely* define an event. Our method of selecting such tetrads is a matter of pure convenience. When we believed in the absolute nature of space, we attributed a definite form to a body for all observers; no matter what kind of co-ordinates they used to define the points

on its surface, it preserved this invariable form which was independent of the co-ordinate system chosen. We no longer accept this, but we now ascribe this invariant property to the world-lines of matter; these have a definite form, and are quite independent of our choice of co-ordinates. The intersections of these world-lines are the coincidences in observation which form almost the entire store of material which we collect and classify, in order to discover sequences in phenomena and to frame laws which summarise such sequences, and the succession and order of such intersections is entirely independent of any numerical method which we adopt so as to distinguish them from one another. We are familiar with so-called "curvilinear" co-ordinates in three-dimensional space. Three families of surfaces are chosen, such that every point in space lies on one and only one member of each family; each member of a family is numbered, and the co-ordinates of a point are the three numbers belonging respectively to the member of the first, of the second, and of the third family on which it lies. Similarly in a four-dimensional continuum we can choose four families of three-dimensional continua, so that every "event" is numbered by the four members, one from each family, to which it is common.

But this breadth of view, although so comprehensive when expression of natural laws is in question, leads to a little perplexity and doubt when we are dealing with measurements and observations of definite phenomena in a definite frame. The relativist cannot "officially" start off by saying: "Choose Cartesian co-ordinates, or polar co-ordinates, etc." He knows that in doing so he is involving himself in difficulties similar to those which would meet a hypothetical two-dimensional being who, while living in a universe that was a surface curved in three dimensions, nevertheless assumed his universe to have the metrical properties of a Euclidean plane. What he must do is to start off with a set of coefficients or "potentials," with which to express the square of an element of a world-line. Any particular set of ten analytical expressions which he chooses is a legitimate set if it obeys the "law of gravitation," i.e., satisfies some tensor law involving the differential coefficients of the $g_{\lambda\mu}$-functions. At present Einstein's law—equation (16)—holds the field. Presumably, if the ten expressions are solutions of it, they possess that particular property which they must have in order to suit the physical nature of our space-time continuum, a property which is quite independent of special frames of reference or systems of co-ordinates. Now although

the Einstein equations (16) are formally symmetrical, it does not follow that the individual expressions for g_{11}, g_{12}, g_{13}, etc., in a legitimate set are symmetrical in the co-ordinates, i.e., that g_{12}, for example, involves the co-ordinates x_1 and x_2 in the same way as g_{34} involves the co-ordinates x_3 and x_4, etc. Bearing this in mind, suppose we have discovered an exact solution, or, at least, a sufficiently approximate one. The question now arises: What do x_1, x_2, x_3, x_4 "mean" in these particular expressions for $g_{\lambda\mu}$? Here we turn gratefully to Euclid and Newton, for we probably notice that our special expression for δs^2 tends towards some expression with which we are already familiar in the restricted theory, when the x_λ approach some limiting values. Perhaps, for example, δs^2 approaches

$$\delta x_\kappa^2 - \delta x_\lambda^2 - \delta x_\mu^2 - \delta x_\nu^2$$

as x_λ, x_μ, x_ν approach infinite values. We at once jump to the conclusion that x_κ is "time," and x_λ, x_μ, x_ν are "rectangular Cartesian co-ordinates." Or it may be that our expression bears a reasonable resemblance to

$$\delta x_\kappa^2 - \delta x_\lambda^2 - x_\lambda^2 \delta x_\mu^2 - x_\lambda^2 \sin^2 x_\mu \delta x_\nu^2,$$

and we assume that x_κ, x_λ, x_μ, x_ν are "time" and "polar co-ordinates." But, although such guesses are extremely serviceable, there is still a small element of doubt in the situation, unless our experimental appliances can be refined beyond the degree of precision as yet attained. This is a point we shall return to when obtaining solutions of the equations.

Another important detail which needs emphasising is concerned with a limitation of the Principle of Equivalence which has been tacitly introduced. The Principle apparently states that the laws of nature which hold in a non-permanent field of force (i.e., in a frame accelerated with reference to a frame which is Galilean over a finite region) will also hold in a permanent field. This statement is certainly invoked when equation (1), which is true in a non-permanent field, is assumed to be the law of motion in a permanent field. But it is denied when equation (15), which holds in a non-permanent field, is assumed not to be true in a permanent, and is replaced by (16). In short, the law of gravitation is a law which limits the principle of equivalence and restricts a statement which as a mere generality would be useless, since without a method of determining what we may call "legitimate" or "natural" potentials, we could not effect a solution of any gravitational problem,

however true in the abstract the Principle of Equivalence might be. Eddington stresses this point in his " Report," and enunciates the Principle in a strict form. We cannot do better than quote him. In Chapter II. (p. 20) he says : " For an infinitesimal region the gravitational force and the force due to a transformation correspond ; but we cannot find any transformation which will remove the gravitational field throughout a finite region. It is like trying to paste a flat sheet of paper on a sphere—the paper can be applied at any point, but as you go away from the point you soon come to a misfit. For this reason it will be desirable to define the exact scope of the Principle of Equivalence. . . . The necessary limitation turns on the number of consecutive points for which gravitational space-time agrees with homaloidal space-time ; in other words, the equivalence will hold only up to a certain order of differential coefficients ;" and later, in Chapter IV. (p. 42) : " The difference between a permanent gravitational field and an artificial one . . . is that in the latter case equation (24. 1) [our equation (16)] is satisfied, whereas in the former the less stringent condition (26. 1) [our equation (15)] is satisfied. These equations determine the second differential coefficients of the $g_{\lambda\mu}$, so that we can make the natural and artificial fields correspond as far as first differential coefficients, but not in the second differential coefficients. We shall therefore state the Principle of Equivalence as follows :

" All laws relating to phenomena in a geometrical field of force, *which depend on the* $g_{\lambda\mu}$ *and their first derivatives*, will also hold in a permanent gravitational field. Laws which depend on the second (or higher) derivatives of the $g_{\lambda\mu}$ will not necessarily apply."

When stated thus the principle covers the adoption of (1) as the equations of motion, for they do not contain second differential coefficients of the potentials.

A considerable simplification of the equations (16) can be introduced if, as Einstein suggests, the co-ordinates are so chosen that $g = -1$. This condition is satisfied in the special Relativity theory ; but, of course, the adoption of it here as an *a posteriori* condition involves no renunciation of the general standpoint, for the condition does not of necessity involve *constant* values for the $g_{\lambda\mu}$-functions. Thus we might find a set of $g_{\lambda\mu}$ which satisfy (16) but do not make g equal to -1. Now by any mathematical transformation we can find another set of $g_{\lambda\mu}$. These would still satisfy Einstein's law, for since $G_{\lambda\mu}$ is a covariant tensor,

$$G_{\lambda\mu}' = b_{\lambda\alpha} b_{\mu\beta} G_{\alpha\beta},$$

and so, if $G_{\lambda\mu} = 0$, it follows that $G_{\lambda\mu}'$ must also be zero. Moreover, since

$$g' = |b|^2 g,$$

it might be possible to choose the transformation, and hence the transformation determinant $|b|$, so that $|b^2|g = -1$, and thus satisfy the added condition. In general equation (16) is

$$\phi_{\lambda\mu} + \psi_{\lambda\mu} = 0,$$

where

$$\phi_{\lambda\mu} = -\partial\{\lambda\mu, \alpha\}/\partial x_\alpha + \{\lambda\alpha, \beta\}\{\mu\beta, \alpha\} \quad . \quad . \quad (18)$$

and

$$\psi_{\lambda\mu} = \partial^2 \log q/\partial x_\lambda \partial x_\mu - \{\lambda\mu, \alpha\}\partial \log q/\partial x_\alpha. \quad . \quad (19)$$

With the condition $g = -1$ the equation becomes

$$\phi_{\lambda\mu} = 0 \quad . \quad . \quad . \quad . \quad (20)$$

It is very necessary, however, to bear in mind that although we have divided the tensor $G_{\lambda\mu}$ into two parts, these parts in general are not individually tensors, and so as to make this point explicit, we have not indicated either part by capital letters. If $g = -1$, $\phi_{\lambda\mu}$ is a tensor, but not otherwise. A very good illustration can be given here of the many pitfalls which wait for the unwary and into which it is only too easy to tumble at a first reading, for, as Eddington remarks, there is "no royal road to Relativity." Seeing that $\psi_{\lambda\mu}$ is the covariant derivative of $\partial \log q/\partial x_\lambda$, which should be a covariant vector, since it is the gradient of a scalar function, why is $\psi_{\lambda\mu}$ not a covariant tensor; and so why should the remaining part, $\phi_{\lambda\mu}$, not be a tensor also, since the whole is a tensor? Now it is true that if

$$\theta(x_1, x_2, x_3, x_4) = \chi(x_1', x_2', x_3', x_4'),$$

then $\partial\theta/\partial x_\lambda$ is a vector whose components in the accented co-ordinates are $\partial\chi/\partial x_\lambda'$. But, in general, g is not equal to g', and so $\log q$ is not in general equal to $\log q'$. In consequence, the gradient of $\log q$ in unaccented co-ordinates is not the same vector as the gradient of $\log q'$ in accented co-ordinates, and the fallacy in the reasoning above is obvious.

The particular choice which Einstein has made for his law of gravitation has an element of arbitrariness in it. This point will be discussed later, but it can be stated now that no other tensor equation can be obtained which involves a tensor of the second rank and contains only first and second derivatives of

the potentials, and the second derivatives in the first degree. Other suggestions have been made, such as

$$g^{\alpha\beta} R_{\alpha\mu\beta}{}^{\kappa} = 0,$$

but it can be shown that this leads to a set of equations equivalent to (16). At present Einstein's law of gravitation appears to be the simplest possible assumption in this tensor analysis, and so far its conclusions agree so well with observation as not to necessitate any resort to a more complicated tensor law.

CHAPTER X.

Electromagnetic Equations in a Gravitational Field.

From this discussion of Einstein's law of gravitation and the limitation thus imposed on the Principle of Equivalence, we now return to the further development of the tensor analysis which is required for the application of the general Principle of Relativity to other natural phenomena.

In Einstein's theory inertia and gravitation are fused into one concept by means of the ten potentials. In Newtonian theory an absolute frame " free from force " is postulated ; in this frame any body travels uniformly in a straight line, or pursues the shortest path ; this is due to its inertia. True, it may be deflected from this path by the mechanical or electromagnetic action of other bodies in the frame, but such deflection is not inherent in the postulated frame. In real frames deflections inherent in the frame are actually observed. These are attributed to a special cause, gravitational force ; and so inertia and gravitation are set over against one another as universal but yet distinct properties of matter. In Einstein's theory they are not so contrasted. In any frame real or hypothetical, all bodies pursue naturally the *path of maximum separation*. This may be curved or not, according to the nature of the frame. Its form is determined *for all bodies* by the values of the $g_{\lambda\mu}$-functions at each point in the frame. Thus Einstein is not compelled to introduce a " gravitational force ; " both gravitation and inertia are contained in the values of the $g_{\lambda\mu}$-functions, and the equations of motion

$$d^2x_\lambda/ds^2 + \{\alpha\beta, \lambda\}dx_\alpha/ds \,.\, dx_\beta/ds = 0 \quad . \quad . \quad (1)$$

require no terms on the right-hand side.

Yet we know that, even in the " inertial frame " of Newton, bodies free from gravitation would not travel straight if they affected each other mechanically or electromagnetically ; neither, we may surmise, will they pursue the path of maximum separation in Einstein's theory in similar circumstances. We

may naturally ask what becomes of the equation of motion in this case. After the course adopted in Chapter VI. there is no difficulty in answering this question. We have seen that the left-hand side of (1) is a contravariant four-vector. Consequently the general equation of motion for a particle in any frame, when affected by non-gravitational influences, is

$$\ddot{x}_\lambda + \{\alpha\beta, \lambda\}\dot{x}_\alpha\dot{x}_\beta = P^\lambda \quad . \quad . \quad . \quad (2)$$

where P^λ is a contravariant vector, whose form depends on those influences, and which we would call the " force-vector." If we proceed as we did in earlier chapters, we will endeavour to show that at all events electromagnetic forces and their activity are capable of being brought into the form of a contravariant four-vector ; but before we can do so, we shall have to discuss the question of the general Relativity of the equations of the electromagnetic field, and so we shall apply ourselves at once to the study of that problem.

It is but natural to turn back to earlier chapters once more for guidance, and we find it to hand. In Chapter VI. we saw that in the restricted theory the expression of the field equations in covariant form was obtained by means of four- and six-vectors, reciprocal six-vectors, and the Lorentzian operator. In Chapters VIII. and IX. we find that these tensors and this operator have their analogues in non-homaloidal continua. Our course is therefore quite clear.*

The field equations were written down in Chapter VII. in the form

$$\partial F_{\lambda\alpha}/\partial x_\alpha = J_\lambda \qquad \text{(4 equations)}$$

and $$\partial F_{\mu\nu}/\partial x_\lambda + \partial F_{\nu\lambda}/\partial x_\mu + \partial F_{\lambda\mu}/\partial x_\nu = 0 \quad \text{(4 equations)} ;$$

or the second tetrad could be expressed as

$$\partial R_{\lambda\alpha}/\partial x_\alpha = 0,$$

where $R_{\lambda\mu}$ is reciprocal to $F_{\lambda\mu}$.

The tensors and vectors involved the components of field and current. One simplifying feature is now absent. We must take account of covariance and contravariance. For example, we can easily see that if we wish to create a current density vector it will be a contravariant vector, and so we shall have to use the symbol J^λ for it and not J_λ. It will be, in fact, the vector $\rho_0 dx_\lambda/ds$, where ρ_0 will be an *invariant* quantity,

* The use of x_4 for imaginary time in Chapters VI. and VII. was only a matter of convenience, and involves no theoretical change.

viz., the density of charge measured in a special frame which will be identified precisely a little later. Recalling the result (10) of Chapter IX., we satisfy the Principle of Relativity if we can throw the equations of the field into the form

$$\partial R_{\mu\nu}/\partial x_\lambda + \partial R_{\nu\lambda}/\partial x_\mu + \partial R_{\lambda\mu}/\partial x_\nu = qJ^\kappa \quad . \quad . \quad (3)$$

and

$$\partial F_{\mu\nu}/\partial x_\lambda + \partial F_{\nu\lambda}/\partial x_\mu + \partial F_{\lambda\mu}/\partial x_\nu = 0 \quad . \quad . \quad (4)$$

where $F_{\lambda\mu}$ and $R_{\lambda\mu}$ are *covariant* six-vectors or "field-tensors."

Equation (3), of course, involves

$$\partial(qJ^a)/\partial x_a = 0,$$

the "equation of continuity."

Taking the fourth component of J^κ, we see that

$$qJ^4 = q\rho_0 dx_4/ds.$$

Let us denote this by ρ. Now consider the group of frames of reference which are Galilean for a given point, and especially that one in which the point is momentarily at rest. If (y_1, y_2, y_3, y_4) are the co-ordinates in this frame of the event (x_1, x_2, x_3, x_4), then by a result in Chapter VIII. (p. 182)

$$q\delta x_1\delta x_2\delta x_3\delta x_4 = \delta y_1\delta y_2\delta y_3\delta y_4$$

and hence

$$\rho_0 q dx_4/ds \,.\, \delta x_1\delta x_2\delta x_3 = \rho_0 dy_4/ds \,.\, \delta y_1\delta y_2\delta y_3$$

i.e.,

$$\rho\delta x_1\delta x_2\delta x_3 = \rho_0\delta y_1\delta y_2\delta y_3,$$

since $dy_4/ds = 1$, for we have chosen a Galilean frame in which the element of volume is at rest. Hence, if we identify ρ_0 as the "proper" charge-density at a point, ρ will be the measure of the charge-density in the gravitational frame where $\rho = q\beta\rho_0 (\beta = dx_4/ds)$, and is the generalised Lorentz factor for volume element moving in the gravitational frame. In fact, since dx_1/dx_4, dx_2/dx_4, dx_3/dx_4 are the velocity components v_1, v_2, v_3,

$$\beta = (g_{11}v_1^2 + \ldots + 2g_{12}v_1v_2 + \ldots + 2g_{14}v_1 + \ldots g_{44})^{-\frac{1}{2}}.$$

This measure of ρ ensures the *invariance* of the charge on a body. Moreover, since

$$qJ^1 = q\beta\rho_0 dx_1/dx_4 = \rho v_1, \text{ etc.},$$

the right-hand sides of (3) become the three components of current-density at a point, and the charge-density at a point as measured in the gravitational frame.

The next step identifies the components of $F_{\lambda\mu}$ and $R_{\lambda u}$ with the components of the field (three-) vectors. In the general

theory of the field for material media, we have four such vectors, the inductions and the intensities, the equations taking the form

$$\left.\begin{aligned} \partial\mathbf{d}/\partial t + \rho\mathbf{v} &= \text{curl}\,\mathbf{h} \\ \text{div}\,\mathbf{d} &= \rho \end{aligned}\right\} \quad . \quad . \quad . \quad (5)$$

$$\left.\begin{aligned} -\partial\mathbf{b}/\partial t &= \text{curl}\,\mathbf{e} \\ \text{div}\,\mathbf{b} &= 0 \end{aligned}\right\} \quad . \quad . \quad . \quad (6)$$

and to obtain solutions, algebraic relations between **b**, **d**, **h**, **e** have to be postulated, usually simple linear relations.

It will be found very convenient to take this form of the equations in a gravitational field even *in vacuo*. It is then easy to identify (5) with (3) and (6) with (4) if the components of the covariant field tensors (antisymmetric, of course) are assumed to be

	$\mu = 1$	$\mu = 2$	$\mu = 3$	$\mu = 4$	
$F_{\lambda\mu} =$	0	$-b_z$	b_y	$-e_x$	$\lambda = 1$
	b_z	0	$-b_x$	$-e_y$	$\lambda = 2$
	$-b_y$	b_x	0	$-e_z$	$\lambda = 3$
	e_x	e_y	e_z	0	$\lambda = 4$
$R_{\lambda\mu} =$	0	$-d_z$	d_y	h_x	$\lambda = 1$
	d_z	0	$-d_x$	h_y	$\lambda = 2$
	$-d_y$	d_x	0	h_z	$\lambda = 3$
	$-h_x$	$-h_y$	$-h_z$	0 *	$\lambda = 4$

where we have introduced x, y, z, t for co-ordinates and time in the gravitational frame, considering them to be alternatives to x_1, x_2, x_3, x_4 as a matter of convenience.

In casting about for "constitutive" relations between **b**, **d**, **e**, **h** as a necessary step towards solution, we must choose them so as to satisfy Relativity and be true to nature in any frame where we can make a test. Referring to Chapter VIII., we pointed out the fact that tensors of different types could be related by "association" with one another. Now $F_{\lambda\mu}$ and $R_{\lambda\mu}$ are both covariant tensors, and could not be so associated; but we can form contravariant tensors from them by "reciprocation." E.g., if $F^{\lambda\mu}$ is the reciprocal of $R_{\lambda\mu}$, we know from Chapter X. that

* It is very necessary to bear in mind the remark immediately after equation (10) of Chapter IX., concerning the replacement of κ, λ, μ, ν by actual digits.

$$qF^{\lambda\mu} = \begin{matrix} 0 & h_z & -h_y & -d_x \\ -h_z & 0 & h_x & -d_y \\ h_y & -h_x & 0 & -d_z \\ d_x & d_y & d_z & 0 \end{matrix}$$

and if $R^{\lambda\mu}$ is reciprocal to $F_{\lambda\mu}$

$$qR^{\lambda\mu} = \begin{matrix} 0 & -e_z & e_y & -b_x \\ e_z & 0 & -e_x & -b_y \\ -e_y & e_x & 0 & -b_z \\ b_x & b_y & b_z & 0 \end{matrix} *$$

Let us now postulate that $R^{\lambda\mu}$ is the contravariant tensor associated with $R_{\lambda\mu}$, so that

$$R^{\lambda\mu} = g^{\lambda\alpha}g^{\mu\beta}R_{\alpha\beta} \quad . \quad . \quad . \quad . \quad (7)$$

This is a set of twelve linear relations between the twelve field vector components. It satisfies Relativity, and it degenerates to the correct relation in a non-gravitational frame. For in that case $g_{11} = g_{22} = g_{32} = -1, g_{44} = 1$, the others being zero; also $g = -1$, and therefore $g^{11} = g^{22} = g^{33} = -1$, $g^{44} = 1$, and the others are zero. Thus it is easily seen that (7) degenerates into the equalities

$$\mathbf{d} = \mathbf{e}$$
$$\mathbf{b} = \mathbf{h}$$

which leads to the usual well-known forms of the equations *in vacuo*.

From a result dealt with at the end of Chapter VIII., we know that equation (7) is consistent with the equation

$$F^{\lambda\mu} = -g^{\lambda\alpha}g^{\mu\beta}F_{\alpha\beta}, \quad . \quad . \quad . \quad (8)$$

so that $F^{\lambda\mu}$ is *minus* the associated tensor of $F_{\lambda\mu}$; and it is easy to see that it likewise degenerates to the familiar form in the absence of gravitation.

A very interesting application of (7) and (8) has recently been made by Silberstein and Rankine.† Let us assume that co-ordinates have been chosen so that $g = -1$, and assume also that the coefficients g_{14}, g_{24}, g_{34} are zero. It follows that $g^{14} = g^{24} = g^{34} = 0$. Hence from equation (7) we find that

* N.B. that the components of these two tensors are not formed from the field vector components themselves, but from their quotients after division by q or $(-g)^{\frac{1}{2}}$.

† "Phil. Mag.," May, 1920, p. 586.

$$-b_x = R^{14} = g^{1\alpha}g^{4\beta}R_{\alpha\beta}$$
$$= g^{44}(g^{11}h_x + g^{12}h_y + g^{13}h_z).$$

Similarly

$$-b_y = g^{44}(g^{21}h_x + g^{22}h_y + g^{23}h_z)$$

and

$$-b_z = g^{44}(g^{31}h_x + g^{32}h_y + g^{33}h_z).$$

That is,

$$b_x = \gamma_{11}h_x + \gamma_{12}h_y + \gamma_{13}h_z \quad . \quad . \quad (9)$$

and two similar equations, where the six coefficients $\gamma_{\lambda\mu}(= \gamma_{\mu\lambda})$ are functions of the tensor components $g_{11}, g_{12}, \ldots, g_{33}, g_{44}$.

In the same way we can show from (8) that

$$d_x = \gamma_{11}e_x + \gamma_{12}e_y + \gamma_{13}e_z \quad . \quad . \quad (10)$$

and two similar equations.

It is known that if we rotate the space axes into the positions occupied by the principal axes of the ellipsoid,

$$\gamma_{11}x^2 + \gamma_{22}y^2 + \gamma_{33}z^2 + 2\gamma_{23}yz + 2\gamma_{31}zx + 2\gamma_{12}xy = 1$$

where γ_{11}, etc., have the values at a point P, then in the neighbourhood of this point the equations (9) and (10) become

$$\left.\begin{aligned} b_x &= \gamma_1 h_x \\ b_y &= \gamma_2 h_y \\ b_z &= \gamma_3 h_z \end{aligned}\right\} \quad (9'). \qquad \left.\begin{aligned} d_x &= \gamma_1 e_x \\ d_y &= \gamma_2 e_y \\ d_z &= \gamma_3 e_z \end{aligned}\right\} \quad (10')$$

the components being now the resolved parts of the vectors along the new axes, and $\gamma_1, \gamma_2, \gamma_3$ being the "principal" tensor components of the *three-dimensional* tensor $\gamma_{\lambda\mu}$. Of course, $\gamma_1, \gamma_2, \gamma_3$ are functions of the $g_{\lambda\mu}$-functions. Putting ρ equal to zero in (5) and (6), let

$$e_x = A_x \sin p\{t + (lx + my + nz)/v\}$$

and two similar equations, and

$$h_x = B_x \sin p\{t + (lx + my + nz)/v\}$$

and two similar equations, so that we are considering the propagation of a plane wave whose frequency is $p/2\pi$, whose wave normal has the direction $-l, -m, -n$, and whose velocity is v. From equation (5) we get

$$\begin{aligned} v\gamma_1 A_x &= mB_z - nB_y & \qquad v\gamma_1 B_x &= nA_y - mA_z \\ v\gamma_2 A_y &= nB_x - lB_z & \qquad v\gamma_2 B_y &= lA_z - nA_x \\ v\gamma_3 A_z &= lB_y - mB_x & \qquad v\gamma_3 B_z &= mA_x - lA_y. \end{aligned}$$

From the first, fifth, and sixth of these it follows that

$$v^2\gamma_1 A_x = m(mA_x - lA_y)/\gamma_3 - n(lA_z - nA_x)/\gamma_2,$$

So $$v^2\gamma_1\gamma_2\gamma_3 A_x = (m^2\gamma_2 + n^2\gamma_3)A_x - l(m\gamma_2 A_y + n\gamma_3 A_z)$$
$$= (l^2\gamma_1 + m^2\gamma_2 + n^2\gamma_3)A_x$$

since $$l\gamma_1 A_x + m\gamma_2 A_y + n\gamma_3 A_z = 0.$$

Hence A_x cancels out and leaves

$$v^2 = (\gamma_1 l^2 + \gamma_2 m^2 + \gamma_3 n^2)/\gamma_1\gamma_2\gamma_3 \quad . \quad . \quad (11)$$

Exactly the same conclusion would be deduced from any other group of three of the equations. The interpretation is that while the velocity of propagation depends on position in the field and direction of propagation (a fact we are already familiar with) it is also independent of the orientation of the plane of polarisation if it is a plane wave. This is a negative conclusion no doubt. Were it not true, a plane wave with its plane of polarisation oblique to the lines of the gravitational field should, if propagated in a direction at right angles to the field, gradually become elliptically-polarised owing to the unequal velocities of propagation of the components of the light vector parallel and perpendicular to the field. A careful search for such an effect was made without result, thus justifying the conclusion drawn from the equations. Apparently the experiment was extremely precise, for had the speeds of the two components of the light vector differed by 10^{-12} of the standard speed, i.e., by ·03 cm. per sec., the ellipticity produced in a path of 40 metres across the Earth's field could have been detected.

Returning now to the general argument, we see that the equations of the electromagnetic field can be thrown into tensor form and so brought under general Relativity. To do so we make use of the covariant field six-vector $F_{\lambda\mu}$, the contravariant field vector $F^{\lambda\mu}$, and the two reciprocal covariant and contravariant field vectors $R_{\lambda\mu}$ and $R^{\lambda\mu}$. With this material a variety of forms are available.

Thus (3) can be written

$$q^{-1}\partial(qF^{\lambda\alpha})/\partial x_\alpha = J^\lambda \quad . \quad . \quad . \quad (3\text{A})$$

which is equivalent to

$$(F^{\lambda\alpha})_\alpha = J^\lambda \quad . \quad . \quad . \quad . \quad (3\text{B})$$

and (4) can be expressed as

$$q^{-1}\partial(qR^{\lambda\alpha})/\partial x_\alpha = 0 \quad . \quad . \quad . \quad (4\text{A})$$

or $$(R^{\lambda\alpha})_\alpha = 0 \quad . \quad . \quad . \quad (4\text{B})$$

Owing to the fact that we can derive a six-vectorial Curl from a covariant four-vector, it is possible to form a vector potential for the electromagnetic field which is a covariant four-vector.

In fact, since $$\operatorname{div} \mathbf{b} = 0,$$
we can write $$\mathbf{b} = \operatorname{curl} \mathbf{a},$$
where $\mathbf{a}$ is a three-dimensional vector potential.

Write $A_1 = a_x$, $A_2 = a_y$, $A_3 = a_z$, $A_4 = -\phi$ where ϕ is the scalar potential of the field. Then A_λ is a covariant four-vector; for assuming it to be so, its Curl ought to be a six-vector, and this is so since

$$\begin{aligned}
\partial A_2/\partial x_3 - \partial A_3/\partial x_2 &= \partial a_y/\partial z - \partial a_z/\partial y \\
&= -b_x \\
&= F_{23} \\
\partial A_1/\partial x_4 - \partial A_4/\partial x_1 &= \partial a_x/\partial t + \partial \phi/\partial x \\
&= -e_x \\
&= F_{14},
\end{aligned}$$

and in general

$$\partial A_\lambda/\partial x_\mu - \partial A_\mu/\partial x_\lambda = F_{\lambda u}.$$

ELECTROMAGNETIC FORCE-ACTIVITY VECTOR.

If we follow the development of the theory in the same manner as in Chapter VI., we consider the vector which is the inner product of J^λ and $F_{\lambda\mu}$. It is, of course, a covariant vector, and we shall denote it by K_λ:

$$K_\lambda = F_{\lambda a} J^a.$$

In terms of the field inductions and intensities, current density and charge density, we have

$$\left.\begin{aligned}
K_1 &= -q^{-1}\rho(e_x + v_y b_z - v_z b_y) \\
K_2 &= -q^{-1}\rho(e_y + v_z b_x - v_x b_z) \\
K_3 &= -q^{-1}\rho(e_z + v_x b_y - v_y b_x) \\
K_4 &= q^{-1}\rho(v_x e_x + v_y e_y + v_z e_z)
\end{aligned}\right\} \quad . \quad . \quad (12)$$

or if we write $\mathbf{k}$ for the three-vector $\rho(\mathbf{e} + [\mathbf{v} \,.\, \mathbf{b}])$,

$$\begin{aligned}
K_1 &= -q^{-1}k_x \\
K_2 &= -q^{-1}k_y \\
K_3 &= -q^{-1}k_z \\
K_4 &= q^{-1}(\mathbf{v} \,.\, \mathbf{k}).^*
\end{aligned}$$

* The difference between the sign occurring here and that in Chapter VI. is due to the change in the sign of the square of an element of separation.

According to the usual interpretation, $\mathbf{k}$ is the electromagnetic force on charged matter per unit volume. Supposing a particle of volume $\delta\tau$ is charged, the electromagnetic force on it is $\mathbf{k}\delta\tau$, or the electromagnetic force-components and the activity are

$$-qK_1\delta\tau,\ -qK_2\delta\tau,\ -qK_3\delta\tau,\ qK_4\delta\tau.$$

Now $q\delta x_1\delta x_2\delta x_3\delta x_4$ is invariant, and therefore $q\beta\delta\tau$ is invariant. Hence if the force on an electrified particle due to the electromagnetic field is $\mathbf{f}$, we see that $\beta f_x, \beta f_y, \beta f_z, -\beta(\mathbf{v}\,.\,\mathbf{f})$ are the components of a *covariant* four-vector. Call it P_λ. Now let us endeavour to construct a possible equation of motion of such a particle in the gravitational field while under electromagnetic force. Let the volume of the particle be $\delta\tau_0$ and μ_0 be its mass density when at rest in Galilean axes. Then an equation such as

$$\mu_0\delta\tau_0(\ddot{x}_\lambda + \{\alpha\beta, \lambda\}\dot{x}_\alpha\dot{x}_\beta) = P_\lambda$$

is impossible, since the left-hand side is contravariant and the right-hand side covariant. But

$$\mu_0\delta\tau_0(\ddot{x}_\lambda + \{\alpha\beta, \lambda\}\dot{x}_\alpha\dot{x}_\beta) = -g^{\lambda\alpha}P_\lambda$$

is a possible equation.

Now

$$\begin{aligned} d/ds &= dx_4/ds\,.\,d/dx_4 \\ &= \beta d/dt. \end{aligned}$$

Therefore the equations become

$$\begin{aligned} d(\mu_0\delta\tau_0\beta dx/dt)/dt + \mu_0\delta\tau_0\beta\phi_1(v_x, v_y, v_z) \\ = -g^{11}f_x - g^{12}f_y - g^{13}f_z + g^{14}(\mathbf{v}\,.\,\mathbf{f}), \text{ etc.} \end{aligned}$$

Now

$$q\beta\delta\tau = \delta\tau_0$$

it follows that

$$\begin{aligned} \beta\mu_0\delta\tau_0 &= q\beta^2\mu_0\delta\tau \\ &= \mu\delta\tau \end{aligned}$$

if

$$\mu = q\beta^2\mu_0.$$

If now μ represents the mass density of the particle in motion in the gravitational frame, the mass of the particle m is related with m_0, its rest mass in Galilean axes, by

$$\begin{aligned} m = \mu\delta\tau &= q\beta^2\mu_0\delta\tau \\ &= \beta\mu_0\delta\tau_0 \\ &= \beta m_0, \end{aligned}$$

and the equations of motion and activity become

$$d(mv_x)/dt + m\phi_1(v_x, v_y, v_z) = -g^{11}f_x - g^{12}f_y - g^{13}f_z + g^{14}(\mathbf{v}\,.\,\mathbf{f})$$

and two similar equations, and

$$dm/dt + m\phi_4(v_x, v_y, v_z) = -g^{41}f_x - g^{42}f_y - g^{43}f_z + g^{44}(\mathbf{v} \cdot \mathbf{f})$$

where $\phi_1, \ldots, \phi_4$ are four quadratic functions of the velocity components whose coefficients are the Christoffel symbols. These equations pass the Relativity test and degenerate to the form arrived at in Chapter VI. for a frame free from gravitation; viz.,

$$\begin{aligned} d(mv_x)/dt &= f_x \\ dm/dt &= (\mathbf{v} \cdot \mathbf{f}). \end{aligned}$$

ELECTROMAGNETIC STRESS-MOMENTUM-ENERGY TENSOR.

The next step is to express K_λ as the generalised Lorentzian of a tensor of the second order. Being guided by the earlier theory in Chapter VI., let us work out the components of the mixed tensor

$$\tfrac{1}{2}(F_{\lambda\alpha}F^{\mu\alpha} - R_{\lambda\alpha}R^{\mu\alpha}),$$

which we will denote by $E_\lambda{}^\mu$:

$$\left.\begin{aligned} E_1{}^1 &= q^{-1}(e_x d_x - \tfrac{1}{2}\Sigma(e_x d_x) + h_x b_x - \tfrac{1}{2}\Sigma(h_x b_x)) \\ E_1{}^2 &= q^{-1}(e_x d_y + h_x b_y) \\ E_1{}^3 &= q^{-1}(e_x d_z + h_x b_z) \\ E_1{}^4 &= -q^{-1}(d_y b_z - d_z b_y) \\ E_2{}^1 &= q^{-1}(e_y d_x + h_y b_x) \\ E_2{}^2 &= q^{-1}(e_y d_y - \tfrac{1}{2}\Sigma(e_x d_x) + h_y b_y - \tfrac{1}{2}\Sigma(h_x b_x) \\ &\;\;\cdot\quad\cdot\quad\cdot \\ &\;\;\cdot\quad\cdot\quad\cdot \\ E_4{}^1 &= q^{-1}(e_y h_z - e_z h_y) \\ &\;\;\cdot\quad\cdot\quad\cdot \\ &\;\;\cdot\quad\cdot\quad\cdot \\ E_4{}^4 &= q^{-1}(\tfrac{1}{2}\Sigma e_x d_x + \tfrac{1}{2}\Sigma h_x b_x) \end{aligned}\right\} \qquad (14)$$

Now let us write

$$\begin{aligned} &t_{xx} \text{ for } (e_x d_x - \tfrac{1}{2}\Sigma(e_x d_x) + h_x b_x - \tfrac{1}{2}\Sigma(h_x b_x)) \\ &t_{xy} \text{ for } e_x d_y + h_x b_y \\ &\quad\cdot\quad\cdot \\ &\quad\cdot\quad\cdot \\ &g_x \text{ for } d_y b_z - d_z b_y \\ &\quad\cdot\quad\cdot \\ &\quad\cdot\quad\cdot \\ &s_x \text{ for } e_y h_z - e_z h_y \\ &\quad\cdot\quad\cdot \\ &\quad\cdot\quad\cdot \\ &\epsilon \text{ for } \tfrac{1}{2}(\Sigma e_x d_x + \Sigma h_x b_x). \end{aligned}$$

In a frame of reference free from gravitation we know that the following equations of momentum and energy-transfer in an electromagnetic field are true:

$$\partial t_{xx}/\partial x + \partial t_{xy}/\partial y + \partial t_{xz}/\partial z - \partial g_x/\partial t = k_x$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\partial s_x/\partial x + \partial s_y/\partial y + \partial s_z/\partial z + \partial \epsilon/\partial t = -(\mathbf{v} \cdot \mathbf{k}),$$

for in that case $g = -1$, $\mathbf{d} = \mathbf{e}$, $\mathbf{b} = \mathbf{h}$.

These are

$$\partial E_\lambda{}^\alpha/\partial x_\alpha = -K_\lambda = -F_{\lambda\alpha}J^\alpha.$$

It follows from the Principle of Relativity that the equations in a gravitational field are

$$(E_\lambda{}^\alpha)_\alpha = -F_{\lambda\alpha}J^\alpha, \quad . \quad . \quad . \quad (15)$$

since both sides are covariant vectors, and each side degenerates into the corresponding side of the equations above, which are true in a special frame.

Using the result (12) of Chapter IX., we can write (15) rather more explicitly as

$$q^{-1}\partial(qE_\lambda{}^\alpha)/\partial x_\alpha - \{\lambda\alpha, \beta\}E_\beta{}^\alpha = -F_{\lambda\alpha}J^\alpha;$$

or, using the values written above in (12) and (14),

$$\left.\begin{array}{l} \partial t_{xx}/\partial x + \partial t_{xy}/\partial y + \partial t_{xz}/\partial z - \partial g_x/\partial t = k_x + \{1\alpha, \beta\}E_\beta{}^\alpha \\ \text{and two similar equations} \\ \partial s_x/\partial x + \partial s_y/\partial y + \partial s_z/\partial z + \partial \epsilon/\partial t = -(\mathbf{v} \cdot \mathbf{k}) + \{4\alpha, \beta\}E_\beta{}^\alpha \end{array}\right\} (16)$$

Interpreting, as usual, t_{xx}, t_{xy}, etc., as components of a stress in an ethereal medium, $\mathbf{g}$ as momentum-density, $\mathbf{s}$ as the Poynting vector or energy flow per unit time per unit area, and ϵ as energy density, we can then interpret

$$-\{1\alpha, \beta\}E_\beta{}^\alpha, \; -\{2\alpha, \beta\}E_\beta{}^\alpha, \; -\{3\alpha, \beta\}E_\beta{}^\alpha$$

as components of the gravitational force exerted on the electromagnetic energy, or as components of the transfer of momentum per unit volume to the electromagnetic field from the gravitational field in one unit of time; $\{4\alpha, \beta\}$ as the transfer of energy per unit volume from the gravitational field to the electromagnetic field in unit time.

Alternative expressions for these transfers are $\pm [\lambda\alpha, \beta]E^{\alpha\beta}$ (which $= \pm \frac{1}{2}E^{\alpha\beta}\partial g_{\alpha\beta}/\partial x_\lambda$), or $\mp \frac{1}{2}E_{\alpha\beta}\partial g^{\alpha\beta}/\partial x_\lambda$,

where $$E^{\lambda\mu} = g^{\lambda\alpha}E_\alpha{}^\mu$$
and $$E_{\lambda\mu} = g_{\mu\alpha}E_\lambda{}^\alpha.$$

If the space axes happen to be those which are " principal " for the linear relations between **d** and **e** and **b** and **h** (see equations (9) and (10)), then

$$\begin{aligned} s_x &= q^{-1}(e_y h_z - e_z h_y) \\ &= (q\gamma_2\gamma_3)^{-1}(d_y b_z - d_z b_y) \\ &= c_1{}^2 g_x \\ s_y &= c_2{}^2 g_y \\ s_z &= c_3{}^2 g_z \end{aligned}$$

where c_1, c_2, c_3 are the velocities of light along the axes (see equation (11)).

These are the customary relations between the energy-stream vector and the momentum-density vector.

It is of interest to express $E_\lambda{}^\mu$ in terms of the two field six-vectors $F_{\lambda\mu}$ and $F^{\lambda\mu}$. This is done as follows:

$$\begin{aligned} R_{1\alpha}R^{1\alpha} &= F^{34}F_{34} + F^{42}F_{42} + F^{23}F_{23} \\ &= \tfrac{1}{2}F^{\alpha\beta}F_{\alpha\beta} - F^{1\alpha}F_{1\alpha} \end{aligned}$$

and hence $F_{1\alpha}F^{1\alpha} - R_{1\alpha}R^{1\alpha} = 2F_{1\alpha}F^{1\alpha} - \tfrac{1}{2}F_{\alpha\beta}F^{\alpha\beta}$.

Also,
$$\begin{aligned} R_{1\alpha}R^{2\alpha} &= R_{13}R^{23} + R_{14}R^{24} \\ &= -F^{24}F_{14} - F^{23}F_{13}. \end{aligned}$$

Therefore $$F_{1\alpha}F^{2\alpha} - R_{1\alpha}R^{2\alpha} = 2F_{1\alpha}F^{2\alpha},$$

and so on.

Hence it is easy to see that

$$E_\lambda{}^\mu = F_{\lambda\alpha}F^{\mu\alpha} - \tfrac{1}{4}g_\lambda{}^\mu F_{\alpha\beta}F^{\alpha\beta}, \qquad (17)$$

remembering the mixed fundamental tensor $g_\lambda{}^\mu$ is equal to unity if $\lambda = \mu$, and to zero if $\lambda \neq \mu$.

CHAPTER XI.

GRAVITATION AND MATTER.

THE DYNAMICS OF CONTINUOUS MEDIA IN A GRAVITATIONAL FIELD.

REMEMBERING the sequence of steps in the development of the theory for homaloidal space-time, we naturally expect that the equation (15) of the previous chapter will give us a clue to the form taken by the equations of momentum and energy transfer in general. This expectation is quite justified, as we shall see; but at the moment we must deal with a question which did not trouble us in the earlier development. We know that in Newtonian theory "mass" plays a double rôle. There is, in the first place, "inertial" mass, the factor which is associated with velocity in the expression for momentum. As a concept it has no direct relation to gravitation, and is defined so as to be free from any confusion with "weight." It is with regard to inertial mass that conservation is assumed to hold in the first instance. But there is also "gravitational" mass whose measure appears in the numerator of the expression which measures the gravitational force between two bodies, the square of their distance apart being the denominator. It is notorious that despite the exhortations of academic literature on the subject, the distinction between the two meanings of the word is a shadowy one to the majority of elementary students, the main reason for this being that in the theory the measures of the two quantities for any body bear to each other a ratio which is independent of the nature, size, and position of the body—a postulate which has been subjected to very searching tests—notably by Eotvös—with confirmatory results. But more advanced students are aware that in the treatment of the gravitational field within matter, the equation of Laplace,

$$\Delta\phi = 0,$$

gives way to the equation of Poisson,

$$\Delta\phi = 4\pi\kappa,$$

where κ is the density of the *gravitational* mass of the matter. Now, as we have seen, Einstein's law of gravitation outside matter,

$$G_{\lambda\mu} = 0,$$

degenerates in a first approximation into the equation of Laplace. It is at once suggested that within matter the potentials must satisfy some other equation, possibly one in which some co-variant tensor replaces the zero of the right-hand side, and that this tensor should be calculable in terms of some quantity or quantities measuring the gravitational influence of the matter ; and if this be so, there arises the question as to the possibility of a close relationship between these quantities and the quantities such as momentum and energy which are covered by the conservation law.

Bearing in mind, therefore, the possible need for a generalisation of Einstein's law of gravitation, let us proceed to deal with the covariance of the equations of momentum and energy transfer in a region where the potentials have general values.

Experience so far suggests that a suitable form for the equations will be possible if a tensor of the mixed type $T_\lambda{}^\mu$ can be constructed out of such material as the density of energy or *inertial* mass, and the density of momentum. In fact, in Chapter VII. we have the degenerate form for such a tensor, and examination of this form shows that the 44-component is ϵ, the energy-density, and if the element of volume surrounding a given point is at rest in a Galilean frame and the matter unstrained, all the components would still further degenerate to zero, except the 44-component which would become ϵ_0, the *rest* energy- (or mass-) density. Now it is not difficult to construct a tensor in any frame which degenerates to ϵ_0 under like circumstances. (In the first instance the tensor is contravariant, but we can then obtain a mixed tensor by association.) Suppose m_0 is the rest-mass of an infinitesimal particle. Then

$$m_0 dx_\lambda/ds \,.\, dx_\mu/ds$$

is a contravariant tensor, since m_0 is an invariant quantity. The sum of such terms is also a tensor, i.e., $\Sigma(m_0 dx_\lambda/ds \,.\, dx_\mu/ds)$ where the summation extends over all the particles within a certain element of *proper* volume $\delta\tau_0$. Now these particles will each have a world-line in the space-time. Let us choose points on these world-lines corresponding to a given value of s ; these points will all lie with a certain four-dimensional element of

volume, and by a natural extension of two and three-dimensional geometry, this element will have a centroid $\bar{x}_1$, $\bar{x}_2$, $\bar{x}_3$, $\bar{x}_4$, so that if $x_1 = \bar{x}_1 + \xi_1$, etc., then

$$\Sigma m_0 \xi_1 = \Sigma m_0 \xi_2 = \Sigma m_0 \xi_3 = \Sigma m_0 \xi_4, = 0$$

and therefore

$$\Sigma(m_0 dx_\lambda/ds \,.\, dx_\mu/ds) = (\Sigma m_0) d\bar{x}_\lambda/ds \,.\, d\bar{x}_\mu/ds + \Sigma m_0 d\xi_\lambda/ds \,.\, d\xi_\mu/ds.$$

The first term on the right-hand side is equal to

$$\beta^2 \mu_0 \delta\tau_0 d\bar{x}_\lambda/d\bar{x}_4 \,.\, d\bar{x}_\mu/d\bar{x}_4$$

where μ_0 is the average density of the rest-mass in the element of proper volume $\delta\tau_0$. On writing $\mu = \beta^2 q \mu_0$, and dropping out the invariant factor $\delta\tau_0$, these terms become

$$q^{-1}\mu\bar{v}_x{}^2,\ q^{-1}\mu\bar{v}_y\bar{v}_x,\ q^{-1}\mu\bar{v}_z\bar{v}_x,\ q^{-1}\mu\bar{v}_x, \text{ etc., } \ldots\ q^{-1}\mu.$$

We take μ as a measure of the density of mass or energy in the gravitational frame, which is greater than the rest-mass by the kinetic energy of the element as a whole, for

$$\mu\delta\tau = \beta^2 q \mu_0 \delta\tau = \beta\mu_0\delta\tau_0.$$

Then $\mu\bar{v}_x$, $\mu\bar{v}_y$, $\mu\bar{v}_z$ are components of momentum-density or energy-stream arising from the motion of the element as a whole.

Treating the second term in the same way, we see that when λ and μ are equal to 1, 2, or 3, the terms are the components of a three-dimensional stress-tensor (of the pressure type) multiplied by q^{-1}. Let us denote this tensor by p_{xx}, p_{xy}, etc. The 14, 24, 34, 41, 42, 43-components are additions to the momentum-density and energy-stream arising from the internal motions of the element, and the 44-component is additional energy-density due to the internal motions (thermal energy) and the existence of pressure within the element of volume. These additional terms are also affected by the factor q^{-1}. Considering μ to represent the total energy- (or inertial mass-) density, $\mathbf{g}$ to represent momentum-density, we see that this contravariant and symmetric tensor is $T^{\lambda\mu}$ where

$$qT^{\lambda\mu} = \begin{array}{r|c|c|c|l}
\mu = 1 & u = 2 & u = 3 & \mu = 4 & \\
\hline
p_{xx} + g_x v_x & p_{yx} + g_y v_x & p_{zx} + g_z v_x & g_x & \lambda = 1 \\
p_{xy} + g_x v_y & p_{yy} + g_y v_y & p_{yy} + g_z v_y & g_y & \lambda = 2 \\
p_{xz} + g_x v_z & p_{yz} + g_y v_z & p_{zz} + g_z v_z & g_z & \lambda = 3 \\
s_x & s_y & s_z & \mu & \lambda = 4
\end{array} \left.\vphantom{\begin{array}{l}1\\2\\3\\4\\5\end{array}}\right\} \qquad (1)$$

(The bar is dropped as no longer necessary.)
The associated mixed tensor is

$$T_\lambda{}^\mu = g_{\lambda\alpha} T^{\mu\alpha} \quad . \quad . \quad . \quad . \quad (2)$$

and there is an associated covariant tensor

$$T_{\lambda\mu} = g_{\lambda\alpha} g_{\mu\beta} T^{\alpha\beta} \quad . \quad . \quad . \quad . \quad (3)$$

In the restricted theory the degenerate form of $T_\lambda{}^\mu$ is

$$-(p_{xx} + g_x v_x), \; -(p_{xy} + g_x v_y), \; -(p_{xz} + g_x v_z), \; -g$$

two similar sets of expressions, and

$$s_x \qquad s_y \qquad s_z \qquad \mu$$

and the equations of transference of momentum and energy to the material system, which are, in this case,

$$\partial p_{xx}/\partial x + \partial p_{xy}/\partial y + \partial p_{xz}/\partial z + Dg_x/\partial t = p_x,$$

two similar equations, and

$$\partial s_x/\partial x + \partial s_y/\partial y + \partial s_z/\partial z = (\mathbf{v} \,.\, \mathbf{p})$$

where $\mathbf{p}$ is the force excluding gravitational force exerted on the matter per unit of volume, can be written

$$\partial T_\lambda{}^\alpha/\partial x_\alpha = -P_\lambda$$

where P_λ is the vector $\beta p_x, \beta p_y, \beta p_z, -\beta(\mathbf{v} \,.\, \mathbf{p})$.
This equation is clearly the degenerate form of

$$(T_\lambda{}^\alpha)_\alpha = -P_\lambda \quad . \quad . \quad . \quad . \quad (4)$$

which is consistent with General Relativity, and so can be taken to be the equation expressing the general laws of mechanics. Alternative forms of (4) are seen by a glance at equations (12) of Chapter IX. to be

$$q^{-1}\partial(qT_\lambda{}^\alpha)/\partial x_\alpha = \{\lambda\alpha, \beta\}T_\beta{}^\alpha - P_\lambda \quad . \quad . \quad (4\text{A})$$

$$\begin{aligned} q^{-1}\partial(qT_\lambda{}^\alpha)/\partial x_\alpha &= [\lambda\alpha, \beta]T^{\alpha\beta} - P_\lambda \\ &= \tfrac{1}{2}T^{\alpha\beta}\partial g_{\alpha\beta}/\partial x_\lambda - P_\lambda \quad . \quad . \quad (4\text{B}) \end{aligned}$$

$$q^{-1}\partial(qT_\lambda{}^\alpha)/\partial x_\alpha = -\tfrac{1}{2}T_{\alpha\beta}\partial g^{\alpha\beta}/\partial x_\lambda - P_\lambda \quad . \quad (4\text{C})$$

We can write equations (4) more explicitly if we introduce these symbols :

$$
\begin{aligned}
\pi_{xx} &= g_{11}p_{xx} + g_{12}p_{yx} + g_{13}p_{zx} \\
\pi_{xy} &= g_{11}p_{xy} + g_{12}p_{yy} + g_{13}p_{zy} \\
&\quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
\pi_{yx} &= g_{21}p_{xx} + g_{22}p_{yx} + g_{23}p_{zx} \\
&\quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
&\quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
\pi_{zz} &= g_{31}p_{xz} + g_{32}p_{yz} + g_{33}p_{zz} \\
\sigma_x &= g_{41}p_{xx} + g_{42}p_{yx} + g_{43}p_{zx} \\
&\quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
\sigma_z &= g_{41}p_{xz} + g_{42}p_{yz} + g_{43}p_{zz} \\
\gamma_x &= g_{11}g_x + g_{12}g_y + g_{13}g_z + g_{14}\mu \\
&\quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
\gamma_z &= g_{31}g_x + g_{32}g_y + g_{33}g_z + g_{34}\mu \\
\nu &= g_{41}g_x + g_{42}g_y + g_{43}g_z + g_{44}\mu.
\end{aligned}
$$

On referring to (4) and remembering that $g_x = s_x = \mu v_x$, we find that they can be written in the form

$$
\left.
\begin{aligned}
&q^{-1}(\partial\pi_{xx}/\partial x + \partial\pi_{xy}/\partial y + \partial\pi_{xz}/\partial z + D\gamma_x/\partial t) \\
&\qquad = \tfrac{1}{2}T^{\alpha\beta}\partial g_{\alpha\beta}/\partial x - \beta p_x \\
&\text{two similar equations, and} \\
&q^{-1}(\partial\sigma_x/\partial x + \partial\sigma_y/\partial y + \partial\sigma_z/\partial z + D\nu/\partial t) \\
&\qquad = \tfrac{1}{2}T^{\alpha\beta}\partial g_{\alpha\beta}/\partial t + \beta(\mathbf{v}\,.\,\mathbf{p})
\end{aligned}
\right\} \quad . \quad (4\text{D})^*
$$

As we have already pointed out, " gravitational force " does not exist as a fundamental concept in the Relativity Theory of Gravitation, but equations (5) show that if we wished to retain the term for convenience of expression, then the components of gravitational force on matter per unit of volume would be measured by

* As usual, the dimensions of the various terms in these equations are rather concealed by the use of the Relativity unit of time. If we wished to introduce the second as the unit of time and express μ in ergs per c.c., g in grammems per c.c., and p_{xx}, etc., in dynes per sq. cm , we would have to alter the quantities g_{11}, g_{24}, g_{34} to numbers c times larger, and g_{44} to a number c^2 times larger. In (5) we would then have to write μ/c for μ in the last term of the expressions for γ_x, γ_y, γ_z, so that this last term would be obviously a momentum-density in grams per c.c. In the expression for ν we would write g_x/c, etc., for g_x, etc., and μ/c^2 for μ, so that ν would be a density in grams per c.c. But actually the dimensions can be shown to be correct in the expressions as written above, if we remember that the velocity of light along the axis of x, for example, is a root of the equation

$$g_{11}(dx/dt)^2 + 2g_{14}dx/dt + g_{44} = 0,$$

so that g_{44}/g_{11} is the square of a velocity; and g_{14}/g_{11} is a velocity, as are g_{24}/g_{11} and g_{34}/g_{11} also.

$$. - \tfrac{1}{2}\beta^{-1}T^{\alpha\beta}\partial g_{\alpha\beta}/\partial x, \text{ etc.} \quad . \quad . \quad . \quad (6)$$

and its activity would be measured by

$$\tfrac{1}{2}\beta^{-1}T^{\alpha\beta}\partial g_{\alpha\beta}/\partial t. \quad . \quad . \quad . \quad . \quad (6\text{A})$$

If the force density **p** which occurs in the equations just dealt with is due to the action of the electromagnetic field on matter, then by reference to Chapter X. we see that

$$P_\lambda = - F_{\lambda\alpha}J^\alpha,$$

and in consequence

$$(T_\lambda{}^\alpha)_\alpha = F_{\lambda\alpha}J^\alpha$$
$$= - (E_\lambda{}^\alpha)_\alpha$$

or

$$(T_\lambda{}^\alpha)_\alpha + (E_\lambda{}^\alpha)_\alpha = 0,$$

which, after all, is consistent with the " material " nature of all energy implicit in the Relativity Theory, and implies that the complete " matter " tensor should embrace the energy, momentum, and stress of the electromagnetic field just as much as the concentrated energy, etc., existing in the minute regions which we term electrons and nuclei. If we desire to be as precise as possible, the electromagnetic field would be defined as that part of the space frame where the **e** and **h** vectors satisfy

$$\left.\begin{array}{l}(F^{\lambda\alpha})_\alpha = 0 \\ (R^{\lambda\alpha})_\alpha = 0\end{array}\right\} \quad . \quad . \quad . \quad . \quad (\text{A})$$

and the gravitational field as that part where the $g_{\lambda\mu}$-components of the fundamental tensor satisfy

$$G_{\lambda\mu} = 0. \quad . \quad . \quad . \quad . \quad (\text{B})$$

" Matter," then, must be that part of the frame where neither (A) nor (B) are true. Instead of (A), we know that in matter

$$(F^{\lambda\alpha})_\alpha = J^\lambda$$
$$(R^{\lambda\alpha})_\alpha = 0$$

may be true where J^λ is defined in terms of the quantity ρ, called density of charge. There are two ways of treating this : either by regarding these parts as of no finite magnitude, but as " singularities in the field " where ρ is infinite, and not concerning ourselves with them at all ; or by giving finite magnitude to them, " smoothing out " the grained structure and attributing a finite value of ρ to each point of it. In the same way we might treat matter as a region throughout which (B) is true except at certain point singularities having definite

" strengths," these strengths being constants introduced in solving for the potentials from (B). We are aware of such treatment in Newtonian theory where matter is regarded as a congeries of points having strengths M_1, M_2, . . . M_n, and the potential at all points except these is given by

$$\sum_{\iota=1}^{n} M_\iota / r_\iota.$$

But just as we can treat Newtonian theory by " smoothing out " these singularities over a finite region, and then introducing Poisson's equation instead of Laplace's, so this method of treatment can be adopted here, and the question arises—What is the analogue of Poisson's equation ?

The Law of Gravitation within Matter.

The answer which has been given by Einstein to this question is obtained by a rather lengthy and involved development of the tensor analysis. The beginner, as he makes his way through the details, is apt to lose the general thread of the argument and forget the main purpose in view, while he is intent on verifying the mathematical truth of each step. So it seems wiser to set out broadly the trend of the argument, filling in the mathematical details later.

The first step consists in the introduction of the function

$$g^{\gamma\epsilon}\{\gamma\alpha, \beta\}\{\epsilon\beta, \alpha\} \quad . \quad . \quad . \quad . \quad (7)$$

which we will denote by the symbol χ. It is obviously homogeneous and quadratic in the quantities $\partial g_{\lambda\mu}/\partial x_\nu$. But by equation (2) of Chapter IX. it appears that it can be transformed into a function homogeneous and quadratic in the quantities $\partial g^{\lambda\mu}/\partial x_\nu$. Also, since each $g_{\lambda\mu}$ is a homogeneous function of degree -1 in the $g^{\lambda\mu}$, it is easy to show that coefficients of the squares and products of the $\partial g^{\lambda\mu}/\partial x_\nu$ in χ are themselves homogeneous functions of the $g^{\lambda\mu}$ of degree -1. In what follows we shall use the symbol $g_\nu{}^{\lambda\mu}$ to represent $\partial g^{\lambda\mu}/\partial x_\nu$. (It is not a tensor, and must not be confused with the covariant derivative $(g^{\lambda\mu})_\nu$, which can be shown to be identically zero by equation (5) of Chapter IX.) Thus χ can be regarded as a function of fifty independent variables, viz., the forty $g_\nu{}^{\lambda\mu}$ and the ten $g^{\lambda\mu}$; and by Euler's theorem on homogeneous functions

$$\left.\begin{aligned} g_\alpha{}^{\beta\gamma}\partial\chi/\partial g_\alpha{}^{\beta\gamma} &= 2\chi \\ g^{\beta\gamma}\partial\chi/\partial g^{\beta\gamma} &= -\chi. \end{aligned}\right\} \quad . \quad . \quad . \quad (8)$$

On giving arbitrary variations to these variables, it appears that

$$\left.\begin{aligned} \partial\chi/\partial g_{\nu}^{\lambda\mu} &= -\{\lambda\mu, \nu\} \\ \partial\chi/\partial g^{\lambda\mu} &= -\{\lambda a, \beta\}\{\mu\beta, a\}. \end{aligned}\right\} \quad . \quad . \quad (9)$$

In the second step, Einstein introduces the sixteen quantities

$$g_{\lambda}^{\mu}\chi - g_{\lambda}^{\beta\gamma}\partial\chi/\partial g_{\mu}^{\beta\gamma} \quad . \quad . \quad . \quad (10)$$

which he denotes by at_{λ}^{μ}. The t_{λ}^{μ} do not constitute a tensor, but, as will appear in the sequel, they enter into covariant equations in a manner similar to tensors, provided a certain restriction is imposed on the co-ordinates. (The multiplier a is a numerical factor which is determined later.) For this reason we do not use the usual capital letter, and we can refer to them as components of a "quasi-tensor" of the mixed type. We have also a quasi-tensor of the covariant type, $g_{\mu a}t_{\lambda}^{a}$, which we will denote by $t_{\lambda\mu}$. If we write t for the contracted quasi-tensor t_{a}^{a}, we have

$$t = 4\chi - 2\chi = 2\chi. \quad . \quad . \quad . \quad (11)$$

It can be proved that

$$a(t_{\lambda}^{\mu} - \tfrac{1}{2}g_{\lambda}^{\mu}t) = g_{\lambda}^{\beta\gamma}\{\beta\gamma, \mu\} \quad . \quad . \quad (12)$$

and

$$a(t_{\lambda\mu} - \tfrac{1}{2}g_{\lambda\mu}t) = g_{\lambda}^{\beta\gamma}[\beta\gamma, \mu]. \quad . \quad . \quad (13)$$

Three equations are now proved involving these quasi-tensors and also the function $\phi_{\lambda\mu}$ defined in equation (18) of Chapter IX., as

$$-\partial\{\lambda\mu, a\}/\partial x_{a} + \{\lambda a, \beta\}\{\mu\beta, a\}.$$

This function is also a quasi-tensor (which acquires full tensor properties in a system of co-ordinates for which $g = -1$), and we can derive from it the associated mixed quasi-tensor $g^{\mu a}\phi_{\lambda a}$ or ϕ_{λ}^{μ}. The three equations are

$$g_{\mu\gamma}\partial(g^{\beta\gamma}\{\lambda\beta, a\})/\partial x_{a} = \tfrac{1}{2}a(t_{\lambda\mu} - \tfrac{1}{2}g_{\lambda\mu}t) - \phi_{\lambda\mu} \quad . \quad (14)$$

$$\partial(g^{\mu\beta}\{\lambda\beta, a\})/\partial x_{a} = \tfrac{1}{2}a(t_{\lambda}^{\mu} - \tfrac{1}{2}g_{\lambda}^{\mu}t) - \phi_{\lambda}^{\mu} \quad . \quad (15)$$

and

$$g_{\lambda}^{\beta\gamma}\phi_{\beta\gamma} = -a\partial t_{\lambda}^{a}/\partial x_{a}. \quad . \quad . \quad . \quad (16)$$

The last step in the analysis produces the equation

$$\begin{aligned} &\tfrac{1}{4}\partial^{2}(g^{a\beta}\partial \log q/\partial x_{\beta})/\partial x_{\lambda}\partial x_{a} \\ &= \tfrac{1}{2}a\partial t_{\lambda}^{a}/\partial x_{a} - (\partial\phi_{\lambda}^{a}/\partial x_{a} - \partial\phi/\partial x_{\lambda}), \end{aligned} \quad (17)$$

where $\phi = g^{a\beta}\phi_{a\beta}$.

On this mathematical material Einstein bases the following train of thought.

Supposing we restrict our co-ordinate system to be such that $g = -1$ everywhere, then the law of gravitation outside matter becomes $\phi_{\lambda\mu} = 0$, and so *outside matter in a system of reference for which* $g = -1$ the following equations are true:

$$g_{\mu\gamma}\partial(g^{\beta\gamma}\{\lambda\beta, a\})/\partial x_a = \tfrac{1}{2}a(t_{\lambda\mu} - \tfrac{1}{2}g_{\lambda\mu}t) \quad . \quad . \quad (14')$$
$$\partial(g^{\mu\beta}\{\lambda\beta, a\})/\partial x_a = \tfrac{1}{2}a(t_\lambda{}^\mu - \tfrac{1}{2}g_\lambda{}^\mu t) \quad . \quad . \quad (15')$$
$$\partial t_\lambda{}^a/\partial x_a = 0. \quad . \quad . \quad . \quad . \quad (16')$$

Equation (16′) is in the form of a conservation law. It is true that $t_\lambda{}^\mu$ is not a tensor, yet the equation (16′) is covariant under the condition $g = -1$; i.e., if we transform to accented co-ordinates for which $g' = -1$, then $\partial t'_\lambda{}^a/\partial x_a' = 0$; but not so if $g' \neq -1$. Just as $T_\lambda{}^\mu$ was a stress-momentum-energy tensor for matter, so we can call this quasi-tensor the stress-momentum-energy quasi-tensor of the gravitational field; and in this field, so long as we deal with an extension in the space-time continuum, which does not contain the world-lines of matter and which is so large that at its boundaries the $g_{\lambda\mu}$ take their constant Galilean values (and in consequence the $t_\lambda{}^\mu$ vanish), then there is conservation of $(t_1{}^4, t_2{}^4, t_3{}^4, t_4{}^4)$, provided, of course, that our co-ordinates are subject to the condition $g = -1$.

It would seem natural, therefore, when dealing with portions of space-time within the boundaries of material world-tubes, to introduce the matter-tensors as well, and replace $t_\lambda{}^\mu$ by $t_\lambda{}^\mu + T_\lambda{}^\mu$, and $t_{\lambda\mu}$ by $t_{\lambda\mu} + T_{\lambda\mu}$.

This would involve, as a conservation law within matter, that the potentials should satisfy

$$\partial(t_\lambda{}^a + T_\lambda{}^a)/\partial x_a = 0 \quad . \quad . \quad . \quad (18)$$

and (14′) and (15′) should be replaced by

$$g_{\mu\gamma}\partial(g^{\beta\gamma}\{\lambda\beta, a\})/\partial x_a = \tfrac{1}{2}a(\{t_{\lambda\mu} + T_{\lambda\mu}\} - \tfrac{1}{2}g_{\lambda\mu}\{t + T\}) \quad . \quad (19)$$
$$\partial(g^{\mu\beta}\{\lambda\beta, a\})/\partial x_a = \tfrac{1}{2}a(\{t_\lambda{}^\mu + T_\lambda{}^\mu\} - \tfrac{1}{2}g_\lambda{}^\mu\{t + T\}), \quad . \quad (20)$$

where T is the invariant quantity $T_a{}^a$, and we are still imposing the condition $g = -1$.

Now it is only necessary to assume that, within matter and subject to the condition $g = -1$, either (19) or (20) is true, for the other one will follow by an inner multiplication; equation (18) can also be deduced from either. Assuming, therefore, that (19) is true within matter, we proceed to modify it somewhat with the help of (15), for, of course, (15) is true anywhere. It is easily seen that from (19) and (15) we obtain

$$\phi_{\lambda\mu} = -\tfrac{1}{2}a(T_{\lambda\mu} - \tfrac{1}{2}g_{\lambda\mu}T) \quad . \quad . \quad . \quad (21)$$

and so Einstein assumes that this is the general law of gravitation for a co-ordinate system in which $g = -1$. To give it a general covariant form is not difficult, for we recall that $\phi_{\lambda\mu}$ is the degenerate form of $G_{\lambda\mu}$ in a system subjected to the restrictive condition $g = -1$. Hence Einstein's complete law of gravitation for any frame is

$$G_{\lambda\mu} = -\tfrac{1}{2}a(T_{\lambda\mu} - \tfrac{1}{2}g_{\lambda\mu}T),$$

which is unconditionally covariant, since it is a linear relation between covariant tensors. The value of a can now be easily obtained by an appeal to the approximate Newtonian theory. Just as in a first approximation $G_{44} = 0$, degenerated into

$$\tfrac{1}{2}\sum_{n=1}^{3}\partial^2 g_{44}/\partial x_n^2 = 0,$$

so

$$G_{44} = -\tfrac{1}{2}a(T_{44} - \tfrac{1}{2}g_{44}T)$$

degenerates into

$$\tfrac{1}{2}\sum_{n=1}^{3}\partial^2 g_{44}/\partial x_n^2 = -\tfrac{1}{4}a\mu_0;$$

for all the components of T_λ^μ and $T_{\lambda\mu}$ vanish except T_4^4 and T_{44}, these being μ_0 in each case, and so is T also equal to μ_0. Moreover, g_{44} is in the limit equal to $1 - 2\psi$, where ψ is the Newtonian potential, and thus

$$\Delta\psi = \tfrac{1}{4}a\mu_0.$$

But this must be Poisson's equation; hence

$$a = 16\pi\kappa,$$

where κ is the Newtonian potential due to a concentrated *erg* of matter at a distance of 1 centimetre. In other words, κ is the astronomical constant $6{\cdot}7 \times 10^{-8}$ divided by the square of c or 9×10^{20}; i.e.,

$$\kappa = 7{\cdot}4 \times 10^{-29}$$

and

$$a = 3{\cdot}74 \times 10^{-27}.$$

It is in this way that Einstein arrives at his general law of gravitation:

$$G_{\lambda\mu} = -8\pi\kappa(T_{\lambda\mu} - \tfrac{1}{2}g_{\lambda\mu}T) \quad . \quad . \quad (22)$$

As regards the standpoint of Relativity, this law is, of course, satisfactory. As regards "truth to nature," it becomes

$$G_{\lambda\mu} = 0$$

outside matter, for there $T_{\lambda\mu} = 0$, and the consequences of this will be discussed in Chapter XIII. In a first approximation we obtain the Laplace-Poisson equation of Newtonian theory.

It will be as well to repeat the reasoning in a somewhat different fashion, which will lead us to a conclusion of extreme importance.

Equations (14)-(17) represent pure mathematical relationships between the various ϕ and t functions, *independent of any law of gravitation.* They are, in fact, in the nature of necessary conditions which no law of gravitation can logically violate. Suppose the law of gravitation is

$$G_{\lambda\mu} = -\tfrac{1}{2}a(T_{\lambda\mu} - \tfrac{1}{2}g_{\lambda\mu}T),$$

where $T_{\lambda\mu}$ is the matter covariant tensor defined earlier in the chapter, and let us limit the co-ordinates so as to satisfy the condition $g = -1$; then we have

$$\tfrac{1}{2}a(T_{\lambda\mu} - \tfrac{1}{2}g_{\lambda\mu}T) = -\phi_{\lambda\mu}.$$

An inner multiplication yields

$$\tfrac{1}{2}a(T_{\lambda}{}^{\mu} - \tfrac{1}{2}g_{\lambda}{}^{\mu}T) = -\phi_{\lambda}{}^{\mu},$$

and by contraction

$$\tfrac{1}{2}aT = \phi,$$

so that

$$\tfrac{1}{2}aT_{\lambda}{}^{\mu} = -(\phi_{\lambda}{}^{\mu} - \tfrac{1}{2}g_{\lambda}{}^{\mu}\phi).$$

Hence

$$\tfrac{1}{2}a\partial T_{\lambda}{}^{\alpha}/\partial x_{\alpha} = -\tfrac{1}{2}a\partial t_{\lambda}{}^{\alpha}/\partial x_{\alpha}$$

by (17), remembering that $\partial \log q/\partial x_{\lambda} = 0$ since g is constant. Therefore

$$\partial(t_{\lambda}{}^{\alpha} + T_{\lambda}{}^{\alpha})/\partial x_{\alpha} = 0.$$

This is (18), and the two just written above are easily seen to be (19) and (20), by reference to (14) and (15), thus justifying the statement that if (19) is assumed to be true as a law of gravitation, (18) can be deduced from it. Equation (18), moreover, is by its form a law of conservation; it should cover the conservative transformation of energy and momentum between matter and the gravitational field. This can be seen still more clearly by eliminating the quasi-tensor $t_{\lambda}{}^{\mu}$ between (18) and (16), when we obtain

$$a\partial T_{\lambda}{}^{\alpha}/\partial x_{\alpha} = g_{\lambda}{}^{\beta\gamma}\phi_{\beta\gamma}.$$

Now referring to (21) and remembering that $g_{\beta\gamma}\partial g^{\beta\gamma}/\partial x_\lambda = 0$, since g, or $g_{\beta\gamma}g^{\beta\gamma}$, is constant, we obtain

$$\partial T_\lambda{}^\alpha/\partial x_\alpha = -\tfrac{1}{2}T_{\beta\gamma}\partial g^{\beta\gamma}/\partial x_\lambda,$$

which is just the equation (4) for this restricted choice of co-ordinates. It therefore appears that Einstein's law of gravitation contains the principle of conservation within it so long as this restriction holds. It is natural to inquire if this restriction is necessary. Supposing, therefore, the co-ordinates are quite general, so that Einstein's complete equation

$$G_{\lambda\mu} = -\tfrac{1}{2}a(T_{\lambda\mu} - \tfrac{1}{2}g_{\lambda\mu}T)$$

is used. By inner multiplication we obtain

$$G_\lambda{}^\mu = g^{\mu\alpha}G_{\lambda\alpha} = -\tfrac{1}{2}a(T_\lambda{}^\mu - \tfrac{1}{2}g_\lambda{}^\mu T) \quad . \quad . \quad (23)$$

and by contraction

$$G = G_\alpha{}^\alpha = g^{\alpha\beta}G_{\alpha\beta} = \tfrac{1}{2}aT.$$

Hence

$$\tfrac{1}{2}aT_\lambda{}^\mu = G_\lambda{}^\mu - \tfrac{1}{2}g_\lambda{}^\mu G.$$

Now we have seen earlier in the chapter that the unconditionally covariant law of transfer between matter and the field has the form

$$(T_\lambda{}^\alpha)_\alpha = 0.$$

Hence if this is to be consistent with the law of gravitation as suggested by Einstein, it follows that

$(G_\lambda{}^\alpha)_\alpha$ is half the contracted derivative of $g_\lambda{}^\mu G$

or $\quad (G_\lambda{}^\alpha)_\alpha = \tfrac{1}{2}\partial G/\partial x_\lambda$. . . . . . . (24)

Now, as a matter of fact, equations (24) are true. But it will, as before, be wiser to postpone the proof of these identical relations until later. It has already been stated that although there are apparently ten equations of gravitation, only six are independent, since there exist four relations between the $G_{\lambda\mu}$ components. Equations (24) are the four relations in question.

Having explained how the structure was raised, it may not be unprofitable to remove the scaffolding and view the result.

As a foundation we have the identical relations, which are independent of any law of gravitation, viz.,

$$(G_\lambda{}^\alpha)_\alpha = \tfrac{1}{2}\partial G/\partial x_\lambda.$$

Einstein suggests as the complete law of gravitation, connecting the potentials at a point with the distribution of material stress, momentum, and energy immediately around this point.

$$G_{\lambda\mu} = -8\pi\kappa(T_{\lambda\mu} - \tfrac{1}{2}g_{\lambda\mu}T)$$

or its equivalent:

$$G_{\lambda\mu} - \tfrac{1}{2}g_{\lambda\mu}G = -8\pi\kappa T_{\lambda\mu} \quad . \quad . \quad . \quad (22\text{A})$$

From this and the above relations we derive the "laws of mechanics":

$$(T_{\lambda}{}^{a})_{a} = 0.$$

In short, Einstein's law is not merely a law of gravitation; it is a *general dynamical principle.* In classical dynamics gravitation stands outside the general system of equations (Lagrange's or Hamilton's), which are entirely independent of any law of gravitation. Only in the identification of gravitational and inertial mass is there any point of contact. In the dynamics of General Relativity the fusion of dynamics and gravitation is complete, and, incidentally, the identification of gravitational and inertial mass is a restricted aspect of the fact that in the calculation of the Einstein tensor $G_{\lambda\mu}$, the inertial mass (energy) and momentum are involved in the tensor $T_{\lambda\mu}$.

We shall now deal with the proof of equations (8)-(16), upon which Einstein's reasoning is based. Before proceeding, the reader should refer for a moment to the equations (2)-(8) of Chapter IX. as they will be frequently employed in what follows. If we vary the independent variables in χ by $\delta g^{\lambda\mu}$ and $\delta g_{\nu}{}^{\lambda\mu}$ we obtain

$$\delta\chi = \{\gamma a, \beta\}\{\epsilon\beta, a\}\delta g^{\gamma\epsilon} + g^{\gamma\epsilon}(\{\gamma a, \beta\}\delta\{\epsilon\beta, a\} + \{\epsilon\beta, a\}\delta\{\gamma a, \beta\}).$$

The second group of terms on the right-hand side

$$\begin{aligned} &= 2g^{\gamma\epsilon}\{\gamma a, \beta\}\delta\{\epsilon\beta, a\} \\ &= 2\{\gamma a, \beta\}\delta(g^{\gamma\epsilon}\{\epsilon\beta, a\}) - 2\{\gamma a, \beta\}\{\epsilon\beta, a\}\delta g^{\gamma\epsilon}. \end{aligned}$$

Hence

$$\delta\chi = -\{\gamma a, \beta\}\{\epsilon\beta, a\}\delta g^{\gamma\epsilon} + 2\{\gamma a, \beta\}\delta(g^{\gamma\epsilon}\{\epsilon\beta, a\}).$$

But

$$2g^{\gamma\epsilon}\{\epsilon\beta, a\} = g^{\gamma\epsilon}g^{a\eta}(\partial g_{\epsilon\eta}/\partial x_{\beta} + \partial g_{\beta\eta}/\partial x_{\epsilon} - \partial g_{\beta\epsilon}/\partial x_{\eta}).$$

In the summation of the previous line it will be found that cancellation of terms gives

$$\begin{aligned}\delta\chi &= -\{\gamma a, \beta\}\{\epsilon\beta, a\}\delta g^{\gamma\epsilon} + \{\gamma a, \beta\}\delta(g^{\gamma\epsilon}g^{a\eta}\partial g_{\epsilon\eta}/\partial x_\beta)\\ &= -\{\gamma a, \beta\}\{\epsilon\beta, a\}\delta g^{\gamma\epsilon} - \{\gamma a, \beta\}\delta(\partial g^{\gamma a}/\partial x_\beta)\\ &= -\{\gamma a, \beta\}\{\epsilon\beta, a\}\delta g^{\gamma\epsilon} - \{\gamma a, \beta\}\delta g_\beta{}^{\gamma a}.\end{aligned}$$

Hence results (9) follow.

Applying (9) to the definition (10), we have

$$at_\lambda{}^\mu = g_\lambda{}^\mu\chi + g_\lambda{}^{\beta\gamma}\{\beta\gamma, \mu\}$$

and by (8)

$$\begin{aligned}at_a{}^a &= g_a{}^a\chi - g_a{}^{\beta\gamma}\partial\chi/\partial g_a{}^{\beta\gamma}\\ &= 4\chi - 2\chi\\ &= 2\chi.\end{aligned}$$

Thus,

$$a(t_\lambda{}^\mu - \tfrac{1}{2}g_\lambda{}^\mu t) = g_\lambda{}^{\beta\gamma}\{\beta\gamma, \mu\},$$

which is (12).

Putting μ equal to a, and effecting an inner multiplication by $g_{\mu a}$, we obtain

$$\begin{aligned}a(t_{\lambda\mu} - \tfrac{1}{2}g_{\lambda\mu}t) &= g_\lambda{}^{\beta\gamma}g_{\mu a}\{\beta\gamma, a\}\\ &= g_\lambda{}^{\beta\gamma}[\beta\gamma, \mu]\end{aligned}$$

which is (13).

In obtaining (14) and (15) we start from

$$\{\lambda\mu, a\} = g_\mu{}^\beta\{\lambda\beta, a\} = g_{\mu\gamma}g^{\beta\gamma}\{\lambda\beta, a\}$$

and differentiate with respect to the co-ordinates so that

$$\partial\{\lambda\mu, a\}/\partial x_a = g_{\mu\gamma}\partial(g^{\beta\gamma}\{\lambda\beta, a\})/\partial x_a + g^{\beta\gamma}\{\lambda\beta, a\}\partial g_{\mu\gamma}/\partial x_a.$$

The second term on the right-hand side is equal to

$$\begin{aligned}&-\{\lambda\beta, a\}g^{\beta\gamma}g_{\gamma\epsilon}g_{\mu\eta}\partial g^{\epsilon\eta}/\partial x_a\\ &= -\{\lambda\beta, a\}g_{\mu\eta}\partial g^{\beta\eta}/\partial x_a\\ &= \{\lambda\beta, a\}g_{\mu\eta}(g^{\beta\gamma}\{\gamma a, \eta\} + g^{\eta\gamma}\{\gamma a, \beta\})\\ &= g^{\beta\gamma}\{\lambda\beta, a\}[a\gamma, \mu] + \{\lambda\beta, a\}\{\mu a, \beta\}\\ &= \tfrac{1}{2}(g^{\gamma\beta}\{\beta\lambda, a\} + g^{a\beta}\{\beta\lambda, \gamma\})[a\gamma, \mu] + \{\lambda\beta, a\}\{\mu a, \beta\}\\ &= -\tfrac{1}{2}g_\lambda{}^{a\gamma}[a\gamma, \mu] + \{\lambda\beta, a\}\{\mu a, \beta\}.\end{aligned}$$

Hence

$$-\phi_{\lambda\mu} = g_{\mu\gamma}\partial(g^{\beta\gamma}\{\lambda\beta, a\})/\partial x_a - \tfrac{1}{2}g_\lambda{}^{\beta\gamma}[\beta\gamma, \mu]. \quad \text{(A)}$$

Putting μ equal to ϵ and employing inner multiplication by $g^{\mu\epsilon}$, we obtain

$$-g^{\mu\epsilon}\phi_{\lambda\epsilon} = \partial(g^{\mu\beta}\{\lambda\beta, a\})/\partial x_a - \tfrac{1}{2}g_\lambda{}^{\beta\gamma}\{\beta\gamma, \mu\}. \quad \text{(B)}$$

Equations (12) and (A) give (14).

,, (13) ,, (B) ,, (15).

We arrive at the third of this group of equations, viz. (16), by writing (12) as

$$at_\lambda{}^\alpha = g_\lambda{}^{\beta\gamma}\{\beta\gamma, \alpha\} + g_\lambda{}^\alpha\chi,$$

differentiating with respect to x_α and observing that $\partial g_\lambda{}^{\beta\gamma}/\partial x_\alpha$ is the same as $\partial g_\alpha{}^{\beta\gamma}/\partial x_\lambda$.

Thus

$$\begin{aligned} a\partial t_\lambda{}^\alpha/\partial x_\alpha &= \partial(g_\lambda{}^{\beta\gamma}\{\beta\gamma, \alpha\})/\partial x_\alpha + \partial\chi/\partial x_\lambda \\ &= \partial(g_\lambda{}^{\beta\gamma}\{\beta\gamma, \alpha\})/\partial x_\alpha + \partial\chi/\partial g^{\beta\gamma} \,.\, \partial g^{\beta\gamma}/\partial x_\lambda \\ &\quad + \partial\chi/\partial g_\alpha{}^{\beta\gamma} \,.\, \partial g_\alpha{}^{\beta\gamma}/\partial x_\lambda \\ &= \partial(g_\lambda{}^{\beta\gamma}\{\beta\gamma, \alpha\})/\partial x_\alpha - \{\beta\epsilon, \eta\}\{\gamma\eta, \epsilon\}g_\lambda{}^{\beta\gamma} \\ &\quad - \{\beta\gamma, \alpha\}\partial g_\alpha{}^{\beta\gamma}/\partial x_\lambda \\ &= g_\lambda{}^{\beta\gamma}(\partial\{\beta\gamma, \alpha\}/\partial x_\alpha - \{\beta\epsilon, \eta\}\{\gamma\eta, \epsilon\}) \\ &= -\, g_\lambda{}^{\beta\gamma}\phi_{\beta\gamma}. \end{aligned}$$

Equation (17) is the result of performing the following operations on (15). Contracting it, multiplying by $\frac{1}{2}g_\lambda{}^\mu$, and subtracting from (15), we have

$$\begin{aligned} &\partial(g^{\mu\beta}\{\lambda\beta, \alpha\})/\partial x_\alpha - \tfrac{1}{2}g_\lambda{}^\mu g^{\beta\gamma}\{\beta\gamma, \alpha\} \\ &= \tfrac{1}{2}at_\lambda{}^\mu - (\phi_\lambda{}^\mu - \tfrac{1}{2}g_\lambda{}^\mu\phi) \quad . \quad . \quad . \quad . \quad \text{(C)} \end{aligned}$$

Now

$$\begin{aligned} g^{\mu\beta}\{\lambda\beta, \alpha\} &= \tfrac{1}{2}g^{\mu\beta}g^{\alpha\epsilon}(\partial g_{\lambda\epsilon}/\partial x_\beta + \partial g_{\beta\epsilon}/\partial x_\lambda - \partial g_{\lambda\beta}/\partial x_\epsilon) \\ &= \tfrac{1}{2}g^{\mu\beta}g^{\alpha\epsilon}\partial g_{\beta\epsilon}/\partial x_\lambda \\ &= -\,\tfrac{1}{2}\partial g^{\mu\alpha}/\partial x_\lambda \end{aligned}$$

and

$$\begin{aligned} g^{\beta\gamma}\{\beta\gamma, \alpha\} &= \tfrac{1}{2}g^{\beta\gamma}g^{\alpha\epsilon}(\partial g_{\beta\epsilon}/\partial x_\gamma + \partial g_{\gamma\epsilon}/\partial x_\beta - \partial g_{\beta\gamma}/\partial x_\epsilon) \\ &= g^{\beta\gamma}g^{\alpha\epsilon}\partial g_{\beta\epsilon}/\partial x_\gamma - \tfrac{1}{2}g^{\alpha\epsilon}g^{\beta\gamma}\partial g_{\beta\gamma}/\partial x_\epsilon \\ &= -\,\partial g^{\alpha\gamma}/\partial x_\epsilon - \tfrac{1}{2}g^{\alpha\epsilon}\partial \log q/\partial x_\epsilon. \end{aligned}$$

Introduce these expressions into the left-hand side of (C), put μ equal to η, operate with $\partial/\partial x_\eta$, and (17) is obtained.

With regard to the very important equations (24), whose proof has also been deferred, it is obvious that if written out *in extenso* the proof would appear to be very complicated and laborious indeed. In his " Space, Time and Gravitation," Eddington says that he doubts " whether anyone has performed the laborious task of verifying these identities by straightforward algebra." Nevertheless, in the mathematical supplement to the French translation of that work he gives an algebraic proof himself, which does not seem to be very involved or tedious. But the proof which follows is due to Jeffery, and appeared in the " Phil. Mag.," **43**, pp. 600-603 (1922). It is quite brief and elegant.

Referring to the ten equations (16) of Chapter IX., and the forty further equations to be obtained by differentiation with respect to x_1, x_2, x_3, x_4 respectively, making fifty equations in all, it is clear that we can obtain in an infinite number of ways values for the coefficients $a_{\lambda\mu}$ and their derivatives $\partial a_{\lambda\mu}/\partial x_\nu$ (eighty quantities in all), which will allow us to prescribe the values of $g_{\lambda\mu}$ and their first differential coefficients *at one definite point-instant* in the world.

Selecting such a definite point-instant for the origin, transform the co-ordinates so that the first differential coefficients of the $g_{\lambda\mu}$ vanish there. In general, second and higher derivatives do not vanish.

The contracted covariant derivative of $G_{\lambda\mu}$ is

$$(G_\lambda{}^\alpha)_\alpha = \partial G_\lambda{}^\alpha/\partial x_\alpha - \{\lambda\alpha,\ \beta\}G_\beta{}^\alpha + \{\beta\alpha,\ \alpha\}G_\lambda{}^\beta$$

which simply becomes at the origin

$$\partial G_\lambda{}^\alpha/\partial x_\alpha$$

which

$$\begin{aligned} &= \partial(g^{\alpha\beta}G_{\lambda\beta})/\partial x_\alpha \\ &= g^{\alpha\beta}\partial G_{\lambda\beta}/\partial x_\alpha. \end{aligned}$$

Substituting the expression for $G_{\lambda\mu}$ from Chapter IX., and omitting terms which vanish on differentiation, we obtain

$$\begin{aligned} (G_\lambda{}^\alpha)_\alpha &= -\tfrac{1}{2}g^{\alpha\beta}g^{\gamma\epsilon}\partial^2(\partial g_{\beta\epsilon}/\partial x_\lambda + \partial g_{\lambda\epsilon}/\partial x_\beta - \partial g_{\lambda\beta}/\partial x_\epsilon)/\partial x_\alpha\partial x_\gamma \\ &\qquad + g^{\alpha\beta}\partial^3 \log q/\partial x_\lambda\partial x_\alpha\partial x_\beta \\ &= -\tfrac{1}{2}g^{\alpha\beta}g^{\gamma\epsilon}\partial^3 g_{\beta\epsilon}/\partial x_\lambda\partial x_\alpha\partial x_\gamma + g^{\alpha\beta}\partial^3 \log q/\partial x_\lambda\partial x_\alpha\partial x_\beta \qquad \text{(D)} \end{aligned}$$

by cancellation of terms on summation. Again,

$$\begin{aligned} \partial G/\partial x_\lambda &= \partial(g^{\alpha\beta}G_{\alpha\beta})/\partial x_\lambda \\ &= -\tfrac{1}{2}g^{\alpha\beta}g^{\gamma\epsilon}\partial^2(\partial g_{\alpha\epsilon}/\partial x_\beta + \partial g_{\beta\epsilon}/\partial x_\alpha - \partial g_{\alpha\beta}/\partial x_\epsilon)/\partial x_\lambda\partial x_\gamma \\ &\qquad + g^{\alpha\beta}\partial^3 \log q/\partial x_\lambda\partial x_\alpha\partial x_\beta. \end{aligned}$$

The first and second terms in the bracket are identical on summation. The third is

$$\tfrac{1}{2}g^{\alpha\beta}g^{\gamma\epsilon}\partial^3 g_{\alpha\beta}/\partial x_\lambda\partial x_\gamma\partial x_\epsilon$$

which, by modifying the dummy suffixes,

$$\begin{aligned} &= \tfrac{1}{2}g^{\alpha\beta}g^{\gamma\epsilon}\partial^3 g_{\gamma\epsilon}/\partial x_\lambda\partial x_\alpha\partial x_\beta \\ &= \tfrac{1}{2}g^{\alpha\beta}\partial^2(g^{\gamma\epsilon}\partial g_{\gamma\epsilon}/\partial x_\alpha)/\partial x_\lambda\partial x_\beta \\ &= g^{\alpha\beta}\partial^3 \log q/\partial x_\lambda\partial x_\alpha\partial x_\beta. \end{aligned}$$

Hence

$$\partial G/\partial x_\lambda = -g^{\alpha\beta}g^{\gamma\epsilon}\partial^3 g_{\beta\epsilon}/\partial x_\lambda \partial x_\alpha \partial x_\gamma + 2g^{\alpha\beta}\partial^3 \log q/\partial x_\lambda \partial x_\alpha \partial x_\beta. \quad \text{(E)}$$

Comparison of (D) and (E) shows that the relations (24) are identically satisfied at the origin in the chosen co-ordinates. As they are tensor relations, they are true at the origin in any system of co-ordinates. But the origin is any arbitrarily chosen point-instant. Hence they are true generally.

CHAPTER XII.

ACTION.

IN the older mechanics the most general form which could be given to the differential equations of motion was first discovered by Lagrange, and a modification of the Lagrangian form was indicated later by Hamilton. Furthermore, it was discovered that dynamical theory could be succinctly expressed in a principle having the widest application. This principle could take one or other of two forms according as one or other of two conditions were imposed—the Principle of Least Action or Hamilton's Principle. The belief gradually gained ground that as knowledge in Physical Science and Chemistry progressed, all natural phenomena would be found to be in the last resort mechanical phenomena occurring in the extremely complicated molecular and atomic systems which constitute matter, and so would come under this principle.

The object of this chapter is, in the first instance, to determine if this principle, generalised if necessary, still holds for Relativity dynamics and, in the second, to see how far physical phenomena are amenable to it when so generalised.

As is well known, complex mechanical systems can be more conveniently described in terms of a number of generalised co-ordinates, such number being the number of the degrees of freedom of the system, rather than in terms of the Cartesian co-ordinates of its separate particles. We represent such generalised co-ordinates by the symbols $q_1, q_2, q_3, \ldots, q_n$. The rates at which these co-ordinates change we shall denote by $r_1, r_2, r_3, \ldots, r_n$. They are, in fact, dq_1/dt, dq_2/dt, etc., the generalised velocities. They are usually indicated by the symbols $\dot{q}_1, \dot{q}_2, \ldots$, etc., in text-books of dynamics; but to avoid confusion we prefer to retain the dotted symbol to indicate differentiation with regard to separation or proper time rather than with regard to the special time of the frame of reference.

The kinetic energy T of the system is a quadratic function

of the velocities r_1, r_2, etc., whose coefficients are in general functions of the co-ordinates q_1, q_2, etc. When the system receives an elementary displacement represented by δq_1, δq_2, etc., its internal potential energy V will suffer a change arising from the work of the forces acting between its mutual parts, and at the same time work will be done on it by external forces involving a transference of energy to the system from its environment, which we represent by

$$Q_1\delta q_1 + Q_2\delta q_2 + \ldots + Q_n\delta q_n,$$

Q_1, Q_2, . . ., Q_n being the generalised components of the external forces acting on the system.

It is then demonstrable that the following equations of motion, due originally to Lagrange, are true:

$$d(\partial T/\partial r_1)/dt - \partial T/\partial q_1 = - \partial V/\partial q_1 + Q_1$$

and $n - 1$ similar equations.

By writing L for $T - V$, and bearing in mind that V depends only on the co-ordinates and not on the velocities, we can express these equations thus:

$$d(\partial L/\partial r_1)/dt - \partial L/\partial q_1 = Q_1, \text{ etc.}$$

L is usually called the Lagrangian function, and is the difference between the kinetic and potential energies of the system.

There is another form for the general equations of motion due to Hamilton. In this the kinetic energy of the system is expressed in terms of the generalised components of momentum. These are the partial differential coefficients of the kinetic energy as expressed above with respect to r_1, r_2, . . ., r_n respectively. Denoting the components of momentum by p_1, p_2, . . ., p_n, we have $p_1 = \partial T/\partial r_1$, $p_2 = \partial T/\partial r_2$, etc. Since T is a quadratic function of r_1, r_2, . . ., r_n it is easily seen that when expressed in terms of the components of momentum it is a quadratic function of p_1, p_2, . . ., p_n, whose coefficients are in general functions of q_1, q_2, . . ., q_n. We denote the kinetic energy when so written by the symbol T_p. It transpires that Hamilton's form for the equations of motion of the system are

$$\partial T_p/\partial p_1 = dq_1/dt = r_1$$

and $n - 1$ similar equations,

and $$\partial T_p/\partial q_1 = - dp_1/dt - \partial V/\partial q_1 + Q_1$$

and $n - 1$ similar equations;

or, writing H for $T_p + V$, we have

$$\partial H/\partial p_1 = dq_1/dt$$

and $n - 1$ similar equations,

$$\partial H/\partial q_1 = - dp_1/dt + Q_1,$$

and $n - 1$ similar equations.

H is generally called the Hamiltonian function, and is equal to the sum of the kinetic and potential energies of the system expressed, of course, as a function of the co-ordinates and *momenta*.

Suppose we follow a certain part of the history of such a system. Such a portion of its history is called a "path;" the beginning of this path is indicated by a set of values for the q_i and r_i, or the q_i and p_i; similarly, the end or any other definite condition of the system at an instant of this history is defined by definite values of the same variables. Such sets of values succeed one another in time in accordance with the equations written above. But it is conceivable that by the introduction of adiabatic constraints (that is, constraints whose reactions would do no work on the system), the system might be made to describe an adjacent path *from the same initial to the same final configuration as before* removed from the first path by differences of co-ordinates and velocities of the first order of magnitude. In addition, the initial velocities in this varied path might be arranged so that the system would obey some further conditions; for instance, it might describe the varied path in the same time, or it might describe it with the same total energy as before.

Following the system in its actual path, we can obtain the integral $\int L dt$ from the initial to the final configuration. Such an integral has the dimensions of a quantity which is the product of energy and time, and is referred to as Action in dynamical theory. We can also obtain the value of this integral for any assigned neighbouring path varied from the actual by means of adiabatic constraints which, of course, introduce no change into the form of the function V as expressed in terms of the co-ordinates. We proceed to state an important theorem concerning this variation of the action thus produced, under the condition that the initial velocities are so adjusted in the

varied path that it is described, from the same initial to the same final condition, *in the same time* as the actual path.

With this condition we can obviously set up a one-one correspondence between the states of the system in its actual history and those in its arbitrarily-varied history. A state in the actual path represented by the co-ordinates and velocities $q_1, q_2, \ldots, q_n, r_1, r_2, \ldots, r_n$ will correspond to a state on the varied path $q_1 + \delta q_1, \ldots, q_n + \delta q_n, r_1 + \delta r_1, \ldots, r_n + \delta r_n$, which is attained after the same lapse of time from the initial instant. The variation symbol δ here refers not to an elementary displacement of the system along either the actual or varied path, but to a *virtual* displacement of the system from an assigned state on the actual path to an adjacent state on the varied path conditioned by an unvaried value of the time. Corresponding to every such assigned state we can calculate the work of the *external* forces on the system if it were actually to receive such a displacement. This virtual work, viz., $\Sigma Q_\iota \delta q_\iota$, we denote by δW. To each state of the actual path would then correspond a definite value of δW if a definite varied path is assigned, and this will also lead to an integral along the path, viz., $\int \delta W dt$. The theorem referred to and known as Hamilton's Principle then asserts that

$$\delta \int L dt + \int \delta W dt = 0, \quad . \quad . \quad . \quad (1)$$

or the variation of the action together with the integral of the virtual work of the external forces is equal to zero. This theorem is obtainable from the Lagrangian equations of motion. On the other hand, if this theorem be taken as the fundamental principle of dynamical theory, the Lagrangian (or any other) form of the equations of motion can be deduced from it.

It will now be our object to investigate what modifications, if any, must be introduced so that this theorem and its deductions may be valid for the wider scope of Relativity dynamics, and, in the first instance, we shall restrict ourselves to the homaloidal world of the earlier theory.

Referring to Chapters III. and VI., we see that the equations of motion of a particle in any reference frame are

$$P_\lambda = d(m dx_\lambda / ds) / ds,$$

where P_λ is the "force-activity" four-vector $\beta f_x, \beta f_y, \beta f_z, \iota\beta(\mathbf{v} \cdot \mathbf{f})$.

In terms of the usual three-dimensional vectors the first three of these are summarised in

$$\mathbf{f} = d(m\mathbf{v})/dt,$$

where, of course, m is not a constant but equal to $m_0(1 - v^2)^{-\frac{1}{2}}$, m_0 being the proper mass. Now let $\delta\mathbf{r}$ represent a virtual displacement to a neighbouring position on a slightly varied path for the particle. Then

$$\begin{aligned}
(\mathbf{f} \, . \, \delta\mathbf{r}) &= (d(m\mathbf{v})/dt \, . \, \delta\mathbf{r}) \\
&= d(m\mathbf{v} \, . \, \delta\mathbf{r})/dt - (m\mathbf{v} \, . \, d\delta\mathbf{r}/dt) \\
&= d(m\mathbf{v} \, . \, \delta\mathbf{r})/dt - (m\mathbf{v} \, . \, \delta d\mathbf{r}/dt) \\
&= d(m\mathbf{v} \, . \, \delta\mathbf{r})/dt - (m\mathbf{v} \, . \, \delta\mathbf{v})
\end{aligned}$$

i.e., $$(\mathbf{f} \, . \, \delta\mathbf{r}) + (m\mathbf{v} \, . \, \delta\mathbf{v}) = d(m\mathbf{v} \, . \, \delta\mathbf{r})/dt.$$

Hence, if we integrate this along the path of the particle, the integral of the left-hand side will be zero, for the integral of the right-hand side is the difference of the values of $(m\mathbf{v} \, . \, \delta\mathbf{r})$ at the final and initial positions which are individually zero since $\delta\mathbf{r}$ is zero at the beginning and end of the varied path.

Now the geometric product $(\mathbf{v} \, . \, \delta\mathbf{v})$ is the ordinary product of the magnitudes of $\mathbf{v}$ and $\delta\mathbf{v}$, i.e., it is $v\delta v$, so

$$\int[mv\delta v + (\mathbf{f} \, . \, \delta\mathbf{r})]dt = 0.$$

In the older mechanics $mv\delta v$ being equal to $m\delta(v^2/2)$, and therefore also to $\delta(mv^2/2)$, we should arrive at once at Hamilton's Theorem for a single particle, which could then be easily extended to any mechanical system regarded as an aggregation of particles. But the variability of mass with velocity, which is a feature of Relativity mechanics, precludes such a step. However, the necessary modification is to hand, for it is not difficult to obtain a function whose differential coefficient with respect to v is equal to the momentum mv. It is, in fact, $-m_0(1 - v^2)^{\frac{1}{2}}$, for

$$\begin{aligned}
d\{-m_0(1 - v^2)^{\frac{1}{2}}\}/dv &= m_0 v(1 - v^2)^{-\frac{1}{2}} \\
&= mv.
\end{aligned}$$

Hence

$$mv\delta v + (\mathbf{f} \, . \, \delta\mathbf{r}) = \delta K + (\mathbf{f} \, . \, \delta\mathbf{r}),$$

where we write K for $-m_0(1 - v^2)^{\frac{1}{2}}$, and so for a single particle

$$\int[\delta K + (\mathbf{f} \, . \, \delta\mathbf{r})]dt = 0.$$

If we now consider a mechanical system consisting of an aggregate of particles which are acted on by external forces

but which do not mutually act on one another, we can derive an equation of action for it by a simple summation of the equations for the individual particles. We represent $\Sigma(\mathbf{f}\,.\,\delta\mathbf{r})$ by δW, and we take the symbol K to mean

$$\Sigma[-m_0(1-v^2)^{\frac{1}{2}}]$$

and so we arrive at

$$\delta\int Kdt + \int\delta W dt = 0 \quad . \quad . \quad . \quad (2)$$

As a first approximation

$$K = \tfrac{1}{2}\Sigma m_0 v^2 - \Sigma m_0$$

and so K approaches in value to a difference between a function $\frac{1}{2}\Sigma m_0 v^2$ the kinetic energy of the older mechanics, and Σm_0, which, according to the newer views, is just a quantity of energy, the internal energy of the individual particles, each one regarded as an isolated system. Of course, K is really equal to $\beta^{-1}\Sigma m_0(\beta - 1) - \Sigma m_0 = \beta^{-1}\Sigma(m - m_0) - \Sigma m_0$ and $\beta^{-1}\Sigma(m - m_0)$ is really a function somewhat less than the kinetic energy of Relativity mechanics.

Apparently the next step in the extension of our views is to consider a system of particles in which interaction between the particles takes place as well as the action of external forces. But, if we do so, we shall be carrying into our new synthesis ideas from the older science which are really foreign to the Relativity standpoint. Action at a distance across intervening space between particles according to some law of distance will not fit into the laws of Relativity dynamics as we already know. So we cannot quietly absorb that part of $\Sigma(\mathbf{f}\,.\,\delta\mathbf{r})$ which refers to "internal" forces into a term such as $-\delta V$, representing a decrease in some potential function. To be logical the step from an aggregate of "inert" particles must be to a system for which, although energy and mass are still highly concentrated within certain small regions, which we may refer to as particles, yet some of the energy and mass is diffused with a small but finite density in the remaining portion of the space occupied by the system. This implies that K is in reality an integral such as $\int\{-\mu_0(1-v^2)^{\frac{1}{2}}\}d\tau_0$ (where μ_0 is now a *proper density*, $d\tau_0$ an element of *proper volume*) rather than a sum of a finite number of terms. K would still be the difference between a term depending on v and approximating to the kinetic energy, and a term $\int\mu_0 d\tau_0$, which would represent intrinsic energy or mass localised mainly but not entirely in

certain minute regions; the "diffused" part of the intrinsic energy would, in a sense, correspond to the mutual potential energy of the particles which arises quite naturally in the mathematical working out of the earlier views.

In the older Physics great success attended the application of equation (1) to all kinds of physical systems in which the view that they were merely aggregates of particles mutually influencing one another by conservative forces was discreetly removed to the background, and in which the object of the mathematical physicist investigating a definite problem was to determine the correct mathematical form to be assigned to the function L in terms of the variables which define the physical state of the given system. In a word, the underlying idea was that the whole behaviour of any physical system could be, as it were, summarised in a definite mathematical expression. It is but natural to inquire if there is any promise of similar success in the newer Physics, supposing an attempt is made to discover a suitable form for the function K (the "kinetic potential") of equation (2).

At once we are faced with a vital question—Does the principle even in the case of purely mechanical systems accord with the postulates of Relativity? We can hardly think otherwise considering that equation (2) has been deduced from equations of motion which themselves agree with these postulates. However, it will be instructive to give a direct proof of the fact. Let us, therefore, investigate the conditions which must hold in order that if we transform from a frame of reference S to another frame S′, then the equation

$$\delta\int K dt + \int \delta W dt = 0$$

transforms into

$$\delta\int K' dt' + \int \delta W' dt' = 0.$$

(Remember we are still postulating the restricted Relativity of a homaloidal world.)

If the system be regarded as a group of particles, each particle will have an individual path in any frame, and to each individual path will correspond a definite world-line. Now taking the frame S, let us consider the path of a definite particle and its neighbouring varied path. These will correspond to a world-line and a varied world-line. The correspondence of the points on the two paths will yield a correspondence of events on the world-lines such that (x_1, x_2, x_3, x_4) corresponds to $(x_1 + \delta x_1, \ldots, x_4 + \delta x_4)$. Now our method of linking the positions on

the paths by means of time, obviously means that the correspondence of events on the world-lines is given by the condition

$$\delta x_4 = 0.$$

Let us indicate points on the actual path by A, B, etc., and the corresponding points on the varied path by Q, R, etc. If now we transfer our thoughts to the frame S′, we are faced with a temporary difficulty arising from our enlarged views concerning simultaneity. The points on the varied path in frame S′ corresponding to A, B, . . . on the actual path are no longer Q, R, . . . ; they are points Y, Z, . . . associated with events on the varied world-line which are related to the events associated with points A, B, . . . by the condition

$$\delta x_4' = 0,$$

and the condition $\delta x_4' = 0$ does not in general agree with the condition $\delta x_4 = 0$. However, the difficulty is easily removed if we remember that since the force-activity four-vector P_λ is related to the momentum-energy four-vector by the equation $P_\lambda = d(m_0 dx_\lambda/ds)/ds = m_0 \ddot{x}_\lambda$, it follows that $P_a \ddot{x}_a$ is zero since $\dot{x}_a \ddot{x}_a$ is zero. Suppose, therefore, x_λ refers to A, $x_\lambda + \delta x_\lambda$ to Q in frame S; let x_λ' and $x_\lambda' + \Delta x_\lambda'$ refer to A and Q in the frame S′; and let $x_\lambda' + \delta x_\lambda'$ refer to Y in frame S′. We have the conditions

$$\delta x_4 = 0$$
$$\delta x_4' = 0$$

but

$$\Delta x_4' \neq 0.$$

By the well-known invariance of geometric products

$$P_a \delta x_a = P_a' \Delta x_a'.$$

Now

$$\Delta x_\lambda' = \delta x_\lambda' + \epsilon \dot{x}_\lambda',$$

where ϵ is some quantity of the first order of magnitude; for Y and Q are on the same path (the varied one), and so the four-vector " direction-cosines " of QY are $\dot{x}_\lambda'$. Hence

$$\begin{aligned} P_a'\Delta x_a' &= P_a'\delta x_a' + \epsilon P_a' \dot{x}_a' \\ &= P_a'\delta x_a'. \end{aligned}$$

Therefore

$$P_a \delta x_a = P_a' \delta x_a'.$$

When translated into terms of frames of reference this becomes, bearing in mind the conditions, $\delta x_4 = 0 = \delta x_4'$

$$\beta(\mathbf{f} \,.\, \delta\mathbf{r}) = \beta'(\mathbf{f}' \,.\, \delta\mathbf{r}').$$

Now the infinitesimal interval of time dt in one frame between A and B is connected with the interval dt' between A and B as measured in the other by the equation

$$dt/\beta = dt'/\beta',$$

for each is the element of proper time. Hence

$$(\mathbf{f} \,.\, \delta\mathbf{r})dt = (\mathbf{f}' \,.\, \delta\mathbf{r}')dt'.$$

Thus since in each frame the total time of the path is unvaried, it follows that

$$\int(\mathbf{f} \,.\, \delta\mathbf{r})dt$$

is invariant.

There is no difficulty now in considering all the particles and proving that

$$\int\delta W dt$$

is invariant.

In order, therefore, that equation (2) may accord with the Relativity postulates, we have the condition that $\int K dt$ should be invariant. Now in so far as K is obtained in the manner suggested by particle dynamics, this is certainly true. For

$$\begin{aligned} K &= -\Sigma m_0 (1 - v^2)^{\frac{1}{2}} \\ &= -\Sigma m_0/\beta, \end{aligned}$$

and therefore

$$\begin{aligned} \int K dt &= -\Sigma\{m_0 \textstyle\int dt/\beta\} \\ &= -\Sigma\{m_0 \textstyle\int ds\} \\ &= -\Sigma\{m_0(s_f - s_i)\}. \end{aligned}$$

It follows that in the extension of the Principle of Stationary Action to physical systems in general an inevitable condition to be satisfied by the kinetic potential chosen for any particular problem is that the Action derived from such a choice must be invariant for all frames of reference.

We are naturally impelled to inquire if a similar proviso has to be established in order to introduce the Principle of Stationary Action into General Relativity. But before proceeding to deal with this very important question it will be instructive to apply the Principle to the case of a general dynamical system and to the case of the electromagnetic field.

For a general dynamical system defined as above it follows from (2) that

$$\int\{\Sigma(\partial L/\partial q_i \,.\, \delta q_i) + \Sigma(\partial L/\partial r_i \,.\, \delta r_i) + \Sigma Q_i\delta q_i\}dt = 0,$$

where we write once more the customary symbol L instead of K.

Also, as $\delta r_i = \delta(dq_i/dt) = d(\delta q_i)/dt$, it follows by steps which will be quite familiar to readers of works on general dynamics that

$$\int\{\Sigma(\partial L/\partial q_i - d(\partial L/\partial r_i)/dt + Q_i)\delta q_i\}dt = 0,$$

and since δq_1, etc., are arbitrary variations, the equations

$$d(\partial L/\partial r_1)/dt - \partial L/\partial q_1 = Q_1, \text{ etc.},$$

are true.

So Lagrange's form of the equations of motion are valid in Relativity dynamics except that the kinetic potential L is no longer the difference between kinetic and potential energies.

We can also arrive at a Hamiltonian form for the equations of motion. Just as for an individual particle

$$d\{-m_0(1 - v^2)^{\frac{1}{2}}\}/dt$$

is equal to the momentum mv, so we can differentiate the function L for any dynamical system with respect to r_1, r_2, etc., and call the quantities so obtained generalised momenta, denoting them by p_1, p_2, etc., as before. Now consider a function which is equal to

$$p_1r_1 + p_2r_2 + \dots + p_nr_n - L,$$

but which is expressed in terms of the *co-ordinates* and *momenta*. Denote it by H. It is easy to apply familiar methods and obtain

$$\partial H/\partial q_1 = - \partial L/\partial q_1$$

and $n - 1$ similar equations, together with

$$\partial H/\partial p_1 = r_1$$

and $n - 1$ similar equations.

From Lagrange's equations it is possible to proceed by steps (which once more will be familiar to most readers) to the following form of the equations of motion:

$$dq_1/dt = \partial H/\partial p_1$$

and $n - 1$ similar equations, together with

$$dp_1/dt = - \partial H/\partial q_1 + Q_1$$

and $n - 1$ similar equations.

It is not difficult by transforming back to Cartesian

co-ordinates to show that H is equal to Σm for the particles of the system, i.e., to the kinetic energy of the system plus the intrinsic energy of its component particles (which latter may be regarded as a constant during many physical processes). Thus the Hamiltonian function H turns out to be, just as in the older dynamics, the total energy of the system expressed in a particular mathematical form. If a system is self-contained and isolated, the principle becomes

$$\delta\int L dt = 0.$$

In extending the Principle of Stationary Action so as to cover the dynamics of continuous media, it is clear that the kinetic potential employed will be expressed as a triple integral whose integrand is the difference between the densities of kinetic energy and energy of strain. In this way it follows that the various elastic solid theories of Light put forward in the nineteenth century can also be summarised in the Action Principle. Finally, as was pointed out by several writers, notably Larmor, the equations of the electromagnetic field can likewise be expressed in a compressed form by equating the variation of a certain integral to zero. This integral will involve not only the co-ordinates and velocities of the charged corpuscles (electrons and nuclei) whose motion constitute the conduction and convection currents, but also some variables in terms of which the field quantities are expressible. The most convenient variables are found to be the components of the three-vector potential and the scalar potential, i.e., the four components of the four-vector potential. In the treatment of a dynamical system not only do the generalised co-ordinates appear in the kinetic potential, but also their differential coefficients with respect to the time; so in the treatment of the field not only do the components of the vector potential appear, but also their differential coefficients with respect to the four space-time co-ordinates. They enter, in fact, into the function involved in the kinetic potential through the six field components, or the components of the field tensor $\mathfrak{F}$ *defined* by

$$\mathfrak{F} = \mathrm{Curl}\,\mathbf{A} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If we now use the symbol L to denote the density of the kinetic potential, we have to consider how the equations of the field can be resumed by the equation

$$\delta\iiiint L(\mathbf{A}, \mathfrak{F}, x, \dot{x})dxdydzdt = 0,$$

under some limiting conditions to be determined, L being a function of $A_1, \ldots, A_4, F_{12}, \ldots, F_{34}, x_1, \ldots, x_4, \dot{x}_1 \ldots \dot{x}_4$.

Since the differential element $dx\,dy\,dz\,dt$ or $dx_1\,dx_2\,dx_3\,dx_4$ of space-time, which we shall in future denote by $d\omega$, is an invariant, it follows that in order to satisfy the postulates of Relativity, the function L must also be invariant.

Suppose now that the paths of the charged material particles are not varied, but that the forms of the functions which express the components of **A** in terms of x_1, x_2, x_3, x_4 are varied infinitesimally from those which actually hold in virtue of the laws of the field, the varied functions being $A_\lambda + \delta A_\lambda$. This variation of **A** will produce a variation in $\mathfrak{F}$ given by

$$\delta\mathfrak{F} = \text{Curl}\ \delta\mathbf{A}.$$

We also assume that the variations vanish at the three-dimensional boundary of the extension in the world through which the integration is effected. (This corresponds to the vanishing of the virtual displacements at the initial and final states of a dynamical system.) Alternatively we can extend the integration to boundaries so far removed that the field components are zero beyond them.

The vanishing of the variation of the action under the conditions stated yields the equation

$$\int\{\partial L/\partial A_\alpha \,.\, \delta A_\alpha + \partial L/\partial F_{\alpha\beta} \,.\, \delta F_{\alpha\beta}\}d\omega = 0;$$

for $\delta x_\lambda = 0 = \delta\dot{x}_\lambda$.

This is equivalent to

$$\int\{\partial L/\partial A_\alpha \,.\, \delta A_\alpha + \partial L/\partial F_{\alpha\beta} \,.\, (\partial\delta A_\beta/\partial x_\alpha - \partial\delta A_\alpha/\partial x_\beta)\}d\omega = 0,$$

interchanging variation and differentiation.

By a small modification of the dummy suffixes this can be written

$$\int\{\partial L/\partial A_\alpha \,.\, \delta A_\alpha + (\partial L/\partial F_{\beta\alpha} - \partial L/\delta F_{\alpha\beta})\partial(\delta A_\alpha)/\partial x_\beta\}d\omega = 0,$$

and this works out to

$$\int\{\partial L/\partial A_\alpha \,.\, \delta A_\alpha + \partial[(\partial L/\partial F_{\beta\alpha} - \partial L/\partial F_{\alpha\beta})\delta A_\alpha]/\partial x_\beta$$
$$- \delta A_\alpha\partial(\partial L/\partial F_{\beta\alpha} - \partial L/\partial F_{\alpha\beta})/\partial x_\beta\}d\omega = 0$$

which reduces to

$$\int\{\partial L/\partial A_\alpha + \partial(\partial L/\partial F_{\alpha\beta} - \partial L/\partial F_{\beta\alpha})/\partial x_\beta\}\delta A_\alpha d\omega = 0,$$

since the triple integral over the boundary vanishes because ∂A_λ is zero at the boundary.

Hence on account of the arbitrary nature of the variations δA_λ, we have

$$\partial L/\partial A_\lambda + \partial(\partial L/\partial F_{\lambda\alpha} - \partial L/\partial F_{\alpha\lambda})/\partial x_\alpha = 0 \quad . \quad . \quad (4)$$

If now the function L has such a form that

$$\partial L/\partial F_{\lambda\mu} = \tfrac{1}{2}F_{\lambda\mu} \quad . \quad . \quad . \quad (5)$$

the equation (4) becomes

$$\partial L/\partial A_\lambda + \partial F_{\lambda\alpha}/\partial x_\alpha = 0,$$

remembering the anti-symmetry of $F_{\lambda\mu}$, and this would agree with field equation of the Lorentz theory,

$$\mathbf{J} = \text{Lor}\ \mathfrak{F},$$

provided

$$\partial L/\partial A_\lambda = -J_\lambda. \quad . \quad . \quad . \quad . \quad (6)$$

It is not difficult to show that equations (5) and (6) are satisfied if L has the form

$$\tfrac{1}{2}(\mathfrak{F}\,.\,\mathfrak{F}) - (\mathbf{A}\,.\,\mathbf{J}) - \mu_0$$

or

$$\tfrac{1}{4}F_{\alpha\beta}F_{\alpha\beta} - A_\alpha J_\alpha - \mu_0 \quad . \quad . \quad . \quad (7)$$

where μ_0 is the proper density of the matter in the system. The last term in (7) would correspond to the part of the action which would exist if there were no field, viz., $-\int\mu_0 d\omega$; for this is equal to $-\iint\mu_0 d\tau_0 ds = -\Sigma\int m_0 ds$, agreeing with a previous result. It is easily seen that if we differentiate the above expression with respect to $F_{\lambda\mu}$ we obtain $\frac{1}{2}F_{\lambda\mu}$, and if with respect to A_λ we obtain J_λ; for in both these differentiations there is no variation of the world-lines of the matter, and so μ_0 and $\mathbf{J}$ are constant.

By imposing a variation under different conditions on the form (7) and equating it to zero, we arrive at the equations of motion for a charged particle as propounded by Lorentz. Suppose, for instance, that with *unvaried values of* $\mathbf{A}$ *and* $\mathfrak{F}$, we subject the stream lines of the particles to a small variation, this will involve a variation of $\mathbf{J}$ and of μ_0, and hence

$$\delta\int L d\omega = -\int(\mathbf{A}\,.\,\delta\mathbf{J})d\omega - \delta\int\mu_0 d\omega,$$

so that if the variation is zero,

$$\int(\mathbf{A}\,.\,\delta\mathbf{J})d\omega = -\delta\int\mu_0 d\omega.$$

Now it is not difficult to show that

$$\delta \mathbf{J} = \text{Lor}\,[\mathbf{J}\,.\,\delta\mathbf{R}]$$

where $\delta\mathbf{R}$ is the vector $(\delta x_1, \delta x_2, \delta x_3, \delta x_4)$, i.e. the virtual displacements in space and (imaginary) time of an element of the current. For instance,

$$\begin{aligned} \delta J_\lambda &= \partial J_\lambda/\partial x_a\,.\,\delta x_a \\ &= \partial J_\lambda/\partial x_a\,.\,\delta x_a - \text{Div}\,\mathbf{J}\,.\,\delta x_\lambda \text{ (since Div } \mathbf{J} = 0) \\ &= \partial(J_\lambda \delta x_a - J_a \delta x_\lambda)/\partial x_a, \end{aligned}$$

which is the λ-component of Lor $[\mathbf{J}\,.\,\delta\mathbf{R}]$.

Therefore

$$\begin{aligned} \int(\mathbf{A}\,.\,\delta\mathbf{J})d\omega &= \int(\mathbf{A}\,.\,\text{Lor}\,[\mathbf{J}\,.\,\delta\mathbf{R}])d\omega \\ &= -\int \text{Div}\,[\mathbf{A}\,.\,[\mathbf{J}\,.\,\delta\mathbf{R}]]d\omega \\ &\quad + \int([\mathbf{J}\,.\,\delta\mathbf{R}]\,.\,\text{Curl}\,\mathbf{A})d\omega.* \end{aligned}$$

The integral of the Divergence can be expressed as an integral over the three-dimensional boundaries (using a four-dimensional analogue of Gauss' Theorem), and this will be zero by reason of the zero values of the variations imposed there.

Hence

$$\begin{aligned} \int(\mathbf{A}\,.\,\delta\mathbf{J})d\omega &= \int(\text{Curl}\,\mathbf{A}\,.\,[\mathbf{J}\,.\,\delta\mathbf{R}])d\omega \\ &= -\int([\mathbf{J}\,.\,\text{Curl}\,\mathbf{A}]\,.\,\delta\mathbf{R})d\omega\,\dagger \\ &= -\int([\mathbf{J}\,.\,\mathfrak{F}]\,.\,\delta\mathbf{R})d\omega. \end{aligned}$$

Now if we were considering the charged matter alone, apart from the field, as a dynamical system, its action would be $-\int\mu_0 d\omega$, and by the proposition established earlier,

$-\,\delta\int\mu_0 d\omega\ +$ virtual work of the external forces on matter $= 0$.

Hence the virtual work of the forces on the matter

$$\begin{aligned} &= \delta\int\mu_0 d\omega \\ &= \int([\mathbf{J}\,.\,\mathfrak{F}]\,.\,\delta\mathbf{R})d\omega \\ &= \int\{\rho(\mathbf{f}\,.\,\delta\mathbf{r}) - \rho(\mathbf{v}\,.\,\mathbf{f})\delta t\}d\tau dt \\ &= \int\{\rho(\mathbf{f}\,.\,[\delta\mathbf{r} - \mathbf{v}t])\}d\tau dt, \end{aligned}$$

where

$$\mathbf{f} = \mathbf{e} + [\mathbf{v}\,.\,\mathbf{h}].$$

* This follows from a theorem which can easily be proved :

$$\text{Div}\,[\mathbf{A}\,.\,\mathfrak{B}] = (\mathfrak{B}\,.\,\text{Curl}\,\mathbf{A}) - (\mathbf{A}\,.\,\text{Lor}\,\mathfrak{B}).$$

† It is easy to prove that

$$(\mathfrak{A}\,.\,[\mathbf{A}\,.\,\mathbf{B}]) = -([\mathbf{A}\,.\,\mathfrak{A}]\,.\,\mathbf{B}) = ([\mathbf{B}\,.\,\mathfrak{A}]\,.\,\mathbf{A}).$$

Now since $\delta\mathbf{r} - \mathbf{v}\delta t$ is the virtual displacement of a charged element to a point on its varied path *for the same value of* t, this result is consistent with the assumption that $\mathbf{f}$ or $\mathbf{e} + [\mathbf{v} \,\dot{:}\, \mathbf{h}]$ is the force exerted by the field on the charged matter per unit of charge.

Hence the value suggested for L yields, by the application of the Action Principle, the electromagnetic equations when one condition is imposed on the variations of the variables occurring in the function, and the mechanical equations when another condition is imposed.

The three terms occurring in (7) can be associated individually with the " field," the " electricity," and the " matter " respectively. The second term can be expressed in another form which brings out its association with the " electricity " very clearly. Thus suppose we divide the charges into infinitesimal amounts de (which are, of course, invariant), and consider their world-lines ; then

$$\begin{aligned}(\mathbf{A} \,.\, \mathbf{J})d\omega &= (A_1\rho v_1 + A_2\rho v_2 + A_3\rho v_3 + \iota A_4\rho)dx_1dx_2dx_3dx_4 \\ &= \iota\rho dx_1dx_2dx_3(A_1dx_1 + \;.\;.\;.\; + A_4dx_4) \\ &= \iota de(A_a dx_a)\end{aligned}$$

where dx_1, dx_2, dx_3, dx_4 are the components of an element of the world-line of de.

Hence

$$\int(\mathbf{A} \,.\, \mathbf{J})d\omega = \iota\iint(\mathbf{A} \,.\, d\mathbf{s})de$$

where the one integration is taken along the world-line of an element of charge de, and the other integration taken over all the charge in the system.

In a similar way we can show that

$$\int\mu_0 d\omega = \iint dsdm_0$$

where ds is an element of the world-line of an element of matter whose proper mass is dm_0, the integration being along these world-lines and over all the matter in the system.

This brief account shows that the Principle of Action, in so far as it was valid in the older Physics is consistent with the Restricted Principle of Relativity. The further steps needed to demonstrate its consistency with General Relativity are due to Einstein himself, and to Hilbert, Lorentz, Klein, and Weyl.

Thus, in so far as the phenomena of the electromagnetic field and the laws of mechanics are concerned, it is extremely

easy to generalise expression (7). One important fact which we have to bear in mind is that in General Relativity the element of space-time $d\omega$ is no longer an invariant, but $qd\omega$, where $q = \sqrt{(-g)}$, is so; in consequence the integrand in the action is no longer an invariant but the product of an invariant by q. Such a product is called a "scalar density" by Weyl, and he also refers to the product of a vector or tensor by q as a "vector-density" or "tensor-density;" the integrals of the products of such densities by $d\omega$ are scalars, vectors, or tensors. The expression which supplants (7) still consists of three terms. The first is now written

$$\tfrac{1}{4}qF_{\alpha\beta}F^{\alpha\beta}.$$

The second term in the action still stands as

$$\int de \int A_\alpha dx_\alpha,$$

and this can be converted into

$$\int qA_\alpha J^\alpha d\omega,$$

for

$$de = \rho_0 d\tau_0 = \rho d\tau,$$

by the invariance of the charge, and so

$$\begin{aligned} dedx_\lambda &= \rho d\tau dx_4 dx_\lambda / dx_4 \\ &= \rho v_\lambda d\omega \\ &= qJ^\lambda d\omega. \end{aligned}$$

Thus the second term in the expression replacing (7) is

$$qA_\alpha J^\alpha d\omega.$$

The third term in the action is still

$$\iint dm_0 ds$$

which will, of course, now involve the $g_{\lambda\mu}$-potentials. This can be expressed in an alternative fashion by writing for $dm_0 ds$

$$dm_0 g_{\alpha\beta} dx_\alpha dx_\beta / ds,$$

which is equal to

$$\begin{aligned} &dm_0 g_{\alpha\beta} \dot{x}_\alpha \dot{x}_\beta ds \\ &= \mu_0 d\tau_0 g_{\alpha\beta} \dot{x}_\alpha \dot{x}_\beta ds \\ &= qg_{\alpha\beta} \mu_0 \dot{x}_\alpha \dot{x}_\beta d\omega \\ &= qg_{\alpha\beta} T^{\alpha\beta} d\omega \\ &= qTd\omega \end{aligned}$$

where $T^{\lambda\mu}$ is the contravariant matter tensor of Chapter XI.

So far, therefore, as we know the expression for the action, we can write it as

$$\tfrac{1}{4}\int qF_{\alpha\beta}F^{\alpha\beta}d\omega - \int de\int A_\alpha dx_\alpha - \iint dm_0 ds \quad . \quad (8)$$

or

$$\int q(\tfrac{1}{4}F_{\alpha\beta}F^{\alpha\beta} - A_\alpha J^\alpha - T)d\omega. \quad . \quad . \quad . \quad (9)$$

As before, if we vary the electromagnetic potentials A_λ, and in consequence the field tensor $F_{\lambda\mu}$, we find, on reference to equation (8) of Chapter X., which gives the relation between $F^{\lambda\mu}$ and $F_{\lambda\mu}$, that the variation of the action is

$$\int q(\tfrac{1}{2}F^{\alpha\beta}\delta F_{\alpha\beta} - J^\alpha\delta A_\alpha)d\omega, \quad . \quad . \quad . \quad (10)$$

provided, of course, the gravitational potentials $g_{\lambda\mu}$ are not varied in functional form, nor the world-lines of matter varied in position.

Suppose, however, that we vary the $g_{\lambda\mu}$-potentials, but keep the A_λ-potentials and the world-lines of the matter unvaried, we find that the integrand of (9) is varied by

$$\tfrac{1}{2}F_{\alpha\beta}\delta(qF^{\alpha\beta}) - qT^{\alpha\beta}\delta g_{\alpha\beta},$$

for, of course, the contravariant field tensor $F^{\lambda\mu}$ depends on the $g_{\lambda\mu}$-potentials. Now

$$\begin{aligned}\delta(qF^{\alpha\beta}) &= F^{\alpha\beta}\delta q - qF_{\gamma\epsilon}\delta(g^{\alpha\gamma}g^{\beta\epsilon})\\ &= \tfrac{1}{2}qF^{\alpha\beta}g^{\gamma\epsilon}\delta g_{\gamma\epsilon} + 2qF_{\gamma\epsilon}g^{\alpha\gamma}g^{\beta\eta}g^{\epsilon\theta}\delta g_{\eta\theta},\end{aligned}$$

by an appeal to equations (6) and (3) of Chapter IX.

Hence the variation in the integrand of (9) is equal to

$$\begin{aligned}&q(F_{\alpha\beta}F_{\gamma\epsilon}g^{\alpha\gamma}g^{\beta\eta}g^{\epsilon\theta}\delta g_{\eta\theta} + \tfrac{1}{4}F_{\alpha\beta}F^{\alpha\beta}g^{\gamma\epsilon}\delta g_{\gamma\epsilon} - T^{\alpha\beta}\delta g_{\alpha\beta})\\ =\ &q(-g^{\beta\eta}F_{\alpha\beta}F^{\alpha\theta}\delta g_{\eta\theta} + \tfrac{1}{4}F_{\alpha\beta}F^{\alpha\beta}g^{\gamma\epsilon}\delta g_{\gamma\epsilon} - T^{\alpha\beta}\delta g_{\alpha\beta})\\ =\ &-q(T^{\alpha\beta} + E^{\alpha\beta})\delta g_{\alpha\beta},\end{aligned}$$

where $E^{\lambda\mu}$ is the *contravariant* stress-momentum-energy tensor of the electromagnetic field introduced in Chapter X.

In consequence, if both types of potentials receive arbitrary functional variations δA_λ and $\delta g_{\lambda\mu}$, it follows that the variation of the expression written in (9) for the action is

$$\int q(\tfrac{1}{2}F^{\alpha\beta}\delta F_{\alpha\beta} - J^\alpha\delta A_\alpha - \tfrac{1}{2}(T^{\alpha\beta} + E^{\alpha\beta})\delta g_{\alpha\beta})d\omega, \quad . \quad (11)$$

where we could make the identification so often signalised in the theory and refer to $T_{\lambda\mu} + E_{\lambda\mu}$ as the complete matter tensor.

If now we wish to use the action function (9) and its variation (11) so as to obtain the electromagnetic and mechanical

equations, we can adopt the following elegant method of procedure, due to Klein.

Conceive that an infinitesimal deformation is applied to the world continuum, which displaces the point-instant (x_1, x_2, x_3, x_4) to (y_1, y_2, y_3, y_4) where

$$y_\lambda = x_\lambda + \epsilon X^\lambda, \quad . \quad . \quad . \quad (12)$$

ϵ being a constant infinitesimal parameter and X^λ being a contravariant vector function of x_1, x_2, x_3, x_4. We also conceive that the covariant vector potential A_λ receives an infinitesimal variation *in its functional form* which transforms it to B_λ in such manner that the invariant product $(\mathbf{A} \, . \, d\mathbf{s})$ for an element of an undisplaced world-line is equal to $(\mathbf{B} \, . \, d\mathbf{s})$ for the corresponding element of the displaced world-line, i.e., that

$$A_a(x)dx_a = B_a(y)dy_a, \quad . \quad . \quad . \quad (13)$$

where we use $A_\lambda(x)$, $B_\lambda(x)$ to denote the values of A_λ and B_λ at the undisplaced world-point (x_1, x_2, x_3, x_4), and $A_\lambda(y)$, $B_\lambda(y)$ to denote values at the displaced world-point (y_1, y_2, y_3, y_4). The variational symbol δ is then to refer to the change at the *undisplaced* world-point, so that

$$\delta A_\lambda = B_\lambda(x) - A_\lambda(x). \quad . \quad . \quad . \quad (14)$$

This variation will involve a variation in $F_{\lambda\mu}$, for since $F_{\lambda\mu}$ is *defined* by

$$F_{\lambda\mu} = \partial A_\mu/\partial x_\lambda - \partial A_\lambda/dx_\mu \quad . \quad . \quad (15)$$

it follows that the new functional form for $F_{\lambda\mu}$, which we shall denote by $H_{\lambda\mu}$, is given by

$$H_{\lambda\mu} = \partial B_\mu/\partial x_\lambda - \partial B_\lambda/\partial x_\mu.$$

Employing (13), it is not difficult to show by means of the four-dimensional analogue of Stoke's Theorem that

$$F_{a\beta}(x)dx_a dx_\beta = H_{a\beta}(y)dy_a dy_\beta. \quad . \quad . \quad (16)$$

As before,

$$\delta F_{\lambda\mu} = H_{\lambda\mu}(x) - F_{\lambda\mu}(x). \quad . \quad . \quad . \quad (17)$$

The varied functional form of the $g_{\lambda\mu}$, which we will denote by $h_{\lambda\mu}$, is to be obtained by a process similar to (17), viz., it has to satisfy the condition that an element of separation in a world-line is not to be altered, i.e.,

$$g_{a\beta}(x)dx_a dx_\beta = h_{a\beta}(y)dy_a dy_\beta \quad . \quad . \quad (18)$$

and $\delta g_{\lambda\mu}$ is then defined by

$$\delta g_{\lambda\mu} = h_{\lambda\mu}(x) - g_{\lambda\mu}(x). \quad . \quad . \quad . \quad (19)$$

We shall denote the variation introduced into the function L by these changes in A_λ, $F_{\lambda\mu}$, $g_{\lambda\mu}$ by δL, but we must be careful to note that the variation of $\int qLd\omega$ *throughout a defined region of the world* is not now $\int\delta(qL)d\omega$, because there has been a deformation of the world continuum itself. The complete variation is, in fact,

$$\int\delta(qL)d\omega + \epsilon\int\partial(qLX^\alpha)/\partial x_\alpha \,.\, d\omega, \quad . \quad . \quad (20)$$

as those familiar with the "divergence" processes used in hydrodynamics and elasticity will recognise.

The second term in (20) arises from the displacement of a small portion of the continuum across the original three-dimensional boundary.

We can calculate the first term in (20) by writing, in accordance with (11),

$$\delta(qL) = \tfrac{1}{2}qF^{\alpha\beta}\delta F_{\alpha\beta} - qJ^\alpha\delta A_\alpha - \tfrac{1}{2}qT^{\alpha\beta}\delta g_{\alpha\beta}$$

(where $T^{\lambda\mu}$ now stands for the complete matter tensor), and replacing $\delta F_{\alpha\beta}$, δA_α, $\delta g_{\alpha\beta}$ by values calculated from (14), (17), (19). Thus by reason of (13)

$$(B_\alpha + \epsilon X^\beta\partial B_\alpha/\partial x_\beta)(dx_\alpha + \epsilon\partial X^\alpha/\partial x_\beta \,.\, dx_\beta) = A_\alpha dx_\alpha,$$

where B_α is, of course, estimated at (x_1, x_2, x_3, x_4).

Hence, neglecting the square of ϵ, we obtain after a little modification of dummy suffixes,

$$(\delta A_\alpha + \epsilon X^\beta\partial B_\alpha/\partial x_\beta + \epsilon B_\beta\partial X^\beta/dx_\alpha)dx_\alpha = 0\,;$$

or, taking the individual multiplier of dx_λ and going to the limit,

$$\delta A_\lambda = -\epsilon(A_\alpha\partial X^\alpha/\partial x_\lambda + X^\alpha\partial A_\lambda/\partial x_\alpha). \quad . \quad . \quad (21)$$

Similarly we can prove that

$$\delta F_{\lambda\mu} = -\epsilon(F_{\lambda\alpha}\partial X^\alpha/\partial x_\lambda + F_{\alpha\mu}\partial X^\alpha/\partial x_\mu + X^\alpha\partial F_{\lambda\mu}/\partial x_\alpha) \quad (22)$$

$$\delta g_{\lambda\mu} = -\epsilon(g_{\lambda\alpha}\partial X^\alpha/\partial x_\lambda + g_{\alpha\mu}\,\partial X^\alpha/\partial x_\mu + X^\alpha\partial g_{\lambda\mu}\,/\partial x_\alpha). \quad (23)$$

If we insert these values in the above we find after a rather tedious but not difficult collection of terms that

$$\delta(qL) = q\epsilon\{(g_{\alpha\gamma}T^{\beta\gamma} - F_{\alpha\gamma}F^{\beta\gamma} + A_\alpha J^\beta)\partial X^\alpha/\partial x_\beta \\ + (\tfrac{1}{2}T^{\beta\gamma}\partial g_{\beta\gamma}/\partial x_\alpha - \tfrac{1}{2}F^{\beta\gamma}\partial F_{\beta\gamma}/\partial x_\alpha + qJ^\beta\partial A_\beta/\partial x_\alpha)X^\alpha\}.$$

The integrand of the second term in (20) can be written as

$$qg_\alpha{}^\beta L\partial X^\alpha/\partial x_\beta + X^\alpha\partial(g_\alpha{}^\beta qL)/\partial x_\beta.$$

So if we write $V_\alpha{}^\beta$ for the mixed tensor

$$T_\alpha{}^\beta - F_{\alpha\gamma}F^{\beta\gamma} + A_\alpha J^\beta + g_\alpha{}^\beta L \quad . \quad . \quad (24)$$

we have the total variation of $\int qLd\omega$, viz.,

$$\epsilon\int d\omega\{qV_\alpha{}^\beta\partial X^\alpha/\partial x_\beta + (\tfrac{1}{2}qT^{\beta\gamma}\partial g_{\beta\gamma}/\partial x_\alpha - \tfrac{1}{2}qF^{\beta\gamma}\partial F_{\beta\gamma}/\partial x_\alpha \\ + qJ^\beta\partial A_\beta/\partial x_\alpha + \partial(g_\alpha{}^\beta qL)/\partial x_\beta)X^\alpha\}.$$

Now if we assume that this variation is zero for any arbitrary displacements of the world-points (subject, of course, to the provisos concerning the variations in A_λ and $g_{\lambda\mu}$ laid down above), i.e., for arbitrary values of X^λ and its derivatives, the individual multipliers of $\partial X^\alpha/\partial x_\beta$ and X^α must be zero. Hence we have

$$V_\lambda{}^\mu = 0\,;$$

or

$$T_\lambda{}^\mu = F_{\lambda\alpha}F^{\mu\alpha} - A_\lambda J^\mu - g_\lambda{}^\mu L. \quad . \quad . \quad (25)$$

To obtain the required equations now is not difficult, for if we assume that the electromagnetic field equations are true (and they can be derived from the Action Principle by a suitably restricted variation, i.e., one in which the $g_{\lambda\mu}$ do not vary, but only the A_λ), we can, by dropping out the terms above which involve the field quantities, and which give a zero result individually because of the field equations, obtain

$$\epsilon\int q[T_\alpha{}^\beta\partial X^\alpha/\partial x_\beta + (\tfrac{1}{2}T^{\beta\gamma}\partial g_{\beta\gamma}/\partial x_\alpha)X^\alpha]d\omega = 0,$$

$$\text{i.e.,} \int\{(-\partial(qT_\alpha{}^\beta)/\partial x_\beta + \tfrac{1}{2}qT^{\beta\gamma}\partial g_{\beta\gamma}/\partial x_\alpha)X^\alpha \\ + \partial(qT_\alpha{}^\beta X^\alpha)/\partial x_\beta\}d\omega = 0.$$

If, as usual, the region of integration is assumed to be bounded by a three-dimensional region where $T_\lambda{}^\mu$ is zero, i.e., where matter does not exist, the divergence part of the integral can be transformed to a boundary integral which vanishes identically. Hence it follows that

$$q^{-1}\partial(qT_\lambda{}^\alpha)/\partial x_\alpha = \tfrac{1}{2}T^{\alpha\beta}\partial g_{\alpha\beta}/\partial x_\lambda,$$

which is just equation (4B) of Chapter XI., because $T_\lambda{}^\mu$ now

includes the electromagnetic field energy-momentum-stress tensor E_λ^μ as well as the " pure " matter tensor.

We have seen in Chapter XI. that the mechanical equations are actually a part of Einstein's law of gravitation—they can be deduced from it. It is but natural to inquire if this law of gravitation can itself be brought within the ambit of the Principle of Stationary Action. If that were so, Physics would have achieved a purpose always implicit in the work of its devotees, it would have, in the words of Weyl, " reduced the totality of natural phenomena to a single physical law." Of course, the accomplishment of such a purpose could not be claimed until we had discovered a mathematical form for the action function in terms of the fundamental variables of the fields agreeing completely with our knowledge of the real world, and no such claim can be made at the moment. We shall, however, conclude this chapter by expounding two methods of bringing the law of gravitation under the Principle of Action, one due to Einstein and one to Lorentz.

What Einstein was able to show was that his law of gravitation within matter could be deduced from the proposition that the integral

$$\int q(G - 8\pi\kappa T)d\omega$$

is stationary or

$$\delta\int q(G - 8\pi\kappa T)d\omega = 0,$$

where T is the scalar invariant of the matter tensor, G the scalar invariant of the Einstein tensor $G_{\lambda\mu}$, regarded as a function of the *contravariant* fundamental tensor $g^{\lambda\mu}$ and its first and second order differential coefficients. (The constant κ is the one introduced in Chapter XI.)

To prove this we begin by putting a limitation on the co-ordinate system which will be removed later; the determinant g is to be equal to minus unity.

Now conceive a finite extension in the world (i.e., the history during a finite time of a limited region of a space frame of reference). We consider the $g^{\lambda\mu}$ and $g_\nu^{\lambda\mu}$ as fifty variables whose values at each point-instant define the condition at that point-instant and its rate of change (spatial and temporal).* In any definite system of reference, the values of these variables at each point pass through a definite sequence of values between two definite instants, provided the initial conditions at the first instant are assigned, i.e., provided we assign the initial

* In what follows $g_\nu^{\lambda\mu}$ stands for $\partial g^{\lambda\mu}/\partial x_\nu$ and $g_{\nu\kappa}^{\lambda\mu}$ for $\partial^2 g^{\lambda\mu}/\partial x_\nu \partial x_\kappa$.

distribution of energy, momentum, and stress. At any point-instant the value of the function χ introduced in (7) of Chapter XI. is calculated, and from it is obtained the value of the integral

$$\iiiint \chi dx_1 dx_2 dx_3 dx_4 \text{ or } \int \chi d\omega$$

throughout the finite extension in space-time which is bounded by a three-dimensional manifold which we will denote by M_3. This boundary will consist of the region of the space-frame $x_4 =$ a constant, k, which is bounded by some definite surface in this space-frame, $\psi(x_1, x_2, x_3) = 0$; the region at the end of the sequence of phenomena in $x_4 = k'$ bounded by the surface with the same equation, and the tubular three-dimensional manifold generated by the world-lines of the points on the two-dimensional bounding surface in the space-frame. The "initial" and "final" instants are, in fact, $x_4 = k$ and $x_4 = k'$.

Now we might conceive that the $g^{\lambda\mu}$ and $g_\nu{}^{\lambda\mu}$ had slightly varied values $g^{\lambda\mu} + \delta g^{\lambda\mu}$ and $g_\nu{}^{\lambda\mu} + \delta g_\nu{}^{\lambda\mu}$ at each point-instant which are not consistent with the law of gravitation, and if these values were introduced into the integral above, a slightly varied value of it would be obtained. The variation would be given by integrating $\delta\chi$ or $-\{\beta\gamma, \alpha\}\delta g_\alpha{}^{\beta\gamma} - \{\beta\epsilon, \theta\}\{\gamma\theta, \epsilon\}\delta g^{\beta\gamma}$ (making use of equations (9) of Chapter XI.).

But $$\delta g_\alpha{}^{\beta\gamma} = \partial(\delta g^{\beta\gamma})/\partial x_\alpha.$$

Hence the first term in the integral can be converted by an integration by parts into

$$\int[\partial\{\beta\gamma, \alpha\}/\partial x_\alpha \,.\, \delta g^{\beta\gamma} - \partial(\{\beta\gamma, \alpha\}\delta g^{\beta\gamma})/\partial x_\alpha]d\omega.$$

By an analogue of Green's theorem, the second term in this can be converted into an integral over the boundary M_3, such as

$$\int n_\alpha\{\beta\gamma, \alpha\}\delta g^{\beta\gamma} dM_3,$$

n_1, n_2, n_3, n_4 being the "direction cosines" of the line at any point of M_3 normal to it.*

* If the equation of M_3 is

$$f(x_1, x_2, x_3, x_4) = 0,$$

then

$$n_1 = \partial f/\partial x_1 \,.\, (\Sigma(\partial f/\partial x_1)^2)^{-\frac{1}{2}}, \text{ etc.}$$

Hence it readily appears that

$$\delta\int\chi d\omega = \int\phi_{\beta\gamma}\delta g^{\beta\gamma}d\omega - \int n_a\{\beta\gamma,\ a\}\delta g^{\beta\gamma}dM_3,$$

where $\phi^{\lambda\mu}$ is the degenerate form of $G_{\lambda\mu}$ when $g = -1$.

But by Einstein's law, if we restrict the co-ordinates to satisfy $g = -1$, the first integral in the right-hand side is equal to

$$-8\pi\kappa\int(T_{\beta\gamma} - \tfrac{1}{2}g_{\beta\gamma}T)\delta g^{\beta\gamma}d\omega.$$

The second term of this is zero, for $g_{\beta\gamma}\delta g^{\beta\gamma}$ is zero provided g is constant. The first term is equal to

$$\begin{aligned} &8\pi\kappa\int[g^{\beta\gamma}\delta T_{\beta\gamma} - \delta(g^{\beta\gamma}T_{\beta\gamma})]d\omega \\ &= 8\pi\kappa\int(g^{\beta\gamma}\delta T_{\beta\gamma} - \delta T)d\omega. \end{aligned}$$

Collecting terms, we see that

$$\delta\int(\chi + 8\pi\kappa T)d\omega = 8\pi\kappa\int g^{\beta\gamma}\delta T_{\beta\gamma}d\omega - \int n_a\{\beta\gamma,\ a\}\delta g^{\beta\gamma}dM_3.$$

Now we suppose (as in the Principle of Least Action in general dynamics) that the initial and final "configurations" are given, i.e., that the varied history starts from the same set of values for the $g^{\lambda\mu}$ at each point within $\psi(x_1, x_2, x_3) = 0$ in the space-frame at $x_4 = k$ as the actual history; and also ends in the same set as the actual history at $x_4 = k'$. We can also regard the surface $\psi(x_1, x_2, x_3) = 0$ as so far removed from matter during the history that the $g^{\lambda\mu}$ have their constant Galilean values there.

As a consequence of these provisos the triple integral vanishes, since $\delta g^{\lambda\mu}$ is zero at all point-instants in M_3. If the quadruple integral is to vanish we must assume the distribution of energy, stress, and momentum to be the same during the varied history as during the actual. This is the analogue of the assumed constancy of the energy in one of the forms by which the Principle of Least Action is expressed. As a matter of fact, constancy of time, $k' - k$, and of energy-stress-momentum are postulated here; and with these conditions satisfied

$$\delta\int(\chi + 8\pi\kappa T)d\omega = 0,$$

or the quadruple integral has a stationary value in the actual history of the gravitational system as in any neighbouring arbitrarily varied history.

It must be observed, however, that the co-ordinates have to be chosen to agree with the condition $g = -1$. This restriction can be removed easily, as follows :

$$\chi - g^{\beta\gamma}\partial\{\beta\gamma, \alpha\}/\partial x_\alpha = \chi - \partial(g^{\beta\gamma}\{\beta\gamma, \alpha\})/\partial x_\alpha + g_\alpha{}^{\beta\gamma}\{\beta\gamma, \alpha\}$$
$$= - \chi - \partial(g^{\beta\gamma}\{\beta\gamma, \alpha\})/\partial x_\alpha$$

by equations (8) and (9) of Chapter XI.

But by the definition of χ, the left-hand side is $g^{\beta\gamma}\phi_{\beta\gamma}$ or ϕ, and so

$\delta\int\phi d\omega = - \delta\int\chi d\omega$ — a boundary integral throughout M_3,

which will vanish as before.

Thus with the same conditions as previously,

$$\delta\int(\phi - 8\pi\kappa T)d\omega = 0.$$

If we now consider the integral

$$\int q(G - 8\pi\kappa T)d\omega,$$

the integrand is the product of two invariants, viz., $G - 8\pi\kappa T$ and $qd\omega$. Hence the value of the integral is an invariant, and so if its variation is zero under arbitrary variation of the $g^{\lambda\mu}$ and $g_\nu{}^{\lambda\mu}$ in any one frame of reference, it must be zero in all. But this is so for frames in which $g = -1$. Hence *with no restriction as to choice of co-ordinates*, but under the " boundary conditions " imposed above, we have

$$\delta\int q(G - 8\pi\kappa T)d\omega = 0 \quad . \quad . \quad . \quad (26)$$

as the most general expression of Einstein's law of gravitation and dynamics.

Lorentz and Hilbert attempt to complete the action function as expressed above in (9) by introducing an additional term to represent the action of the gravitational field. In (9) we have a term involving A_λ representing the " substance action " of the electricity, a term involving linear functions of the first derivatives of A_λ to represent the " field-action " of the electricity, and a term involving $g_{\lambda\mu}$ representing the substance action of mass. Apparently all that is needed to complete the function and round everything off with mathematical elegance is to add a fourth term to represent the " field-action " of the matter, i.e., gravitation; such a term should involve linear functions of the first derivatives of the $g_{\lambda\mu}$, the most natural functions to choose being the $[\lambda\mu, \nu]$ or $\{\lambda\mu, \nu\}$ indexes of Christoffel. But, unfortunately, we know of no *invariants* which involve only the $g_{\lambda\mu}$ or $g^{\lambda\mu}$ and their *first* derivatives. We have, of course, the contracted Riemann tensor $G_{\lambda\mu}$ and its scalar invariant $G(= g^{\alpha\beta}G_{\alpha\beta})$; but these involve the second derivatives of $g_{\lambda\mu}$ linearly.

However, we can make shift by the well-worn device of introducing a partial integration and discarding a divergence integral which can be reduced to a boundary integral vanishing at the limits of the region of integration.

Stated formally, the action is now written

$$\int q(L_1 + L_2 + L_3 + L_4)d\omega.$$

(*a*) L_1 is the substance action of the electricity, and is

$$-A_\alpha J^\alpha.$$

(*b*) L_2 is the field action of the electricity, and is

$$\tfrac{1}{4}F_{\alpha\beta}F^{\alpha\beta}.$$

(*c*) L_3 is the substance action of matter, and is

$$-g_{\alpha\beta}T^{\alpha\beta}.$$

(*d*) On referring to (17) of Chapter IX., it will be seen that we can divide the tensor $G_{\lambda\mu}$ into two parts:

$$\{\lambda\alpha, \beta\}\{\mu\beta, \alpha\} - \{\lambda\mu, \alpha\}\{\alpha\beta, \beta\}$$

and

$$\partial\{\lambda\alpha, \alpha\}/\partial x_\mu - \partial\{\lambda\mu, \alpha\}/\partial x_\alpha,$$

the first part only involving the first derivatives of the $g_{\lambda\mu}$ (or $g^{\lambda\mu}$), the second part involving second derivatives. So the scalar invariant can also be split into two parts:

$$g^{\gamma\epsilon}\{\gamma\alpha, \beta\}\{\epsilon\beta, \alpha\} - g^{\gamma\epsilon}\{\gamma\epsilon, \alpha\}\{\alpha\beta, \beta\}$$

and

$$g^{\gamma\epsilon}\partial\{\gamma\alpha, \alpha\}/\partial x_\epsilon - g^{\gamma\epsilon}\partial\{\gamma\epsilon, \alpha\}/\partial x_\alpha.$$

We take L_4 to be a multiple of the first part, so that

$$L_4 = kg^{\gamma\epsilon}(\{\gamma\alpha, \beta\}\{\epsilon\beta, \alpha\} - \{\gamma\epsilon, \alpha\}\{\alpha\beta, \beta\})$$

where k is some numerical constant, so that L_4 involves only the functions $\{\lambda\mu, \nu\}$, i.e., the derivatives of $g_{\lambda\mu}$ to the first order, but no higher. But L_4 is not an invariant.*

However, if we consider the variation of the integral

$$\int qL_4 d\omega,$$

it can be shown by the device of integration by parts now familiar to the reader that it is equal to

$$k\int qK_{\alpha\beta}\delta g^{\alpha\beta}d\omega$$

* The reader will observe that this splitting of $G_{\lambda\mu}$ and G is quite different from the splitting into $\phi_{\lambda\mu}$ and $\psi_{\lambda\mu}$. In fact, L_4 is not invariant even for a co-ordinate system satisfying the condition $g = -1$.

where $K_{\lambda\mu}$ is the tensor

$$\tfrac{1}{2}g_{\lambda\mu}G - G_{\lambda\mu},$$

so that although the term $\int qL_4 d\omega$ of the proposed action function is not invariant, its variation is, and, after all, although the desire for complete mathematical elegance is not satisfied, the necessities of the physical principle are.

It will now be obvious to the reader by a repetition of former steps that

1° if we vary the electromagnetic potentials A_λ alone and put the variation equal to zero we obtain the electromagnetic field equations.

2° if we vary the gravitational potentials alone and annul the variation we shall obtain

$$K_{\lambda\mu} - T_{\lambda\mu} - E_{\lambda\mu} = 0.$$

A reference to (22) or (22A) of Chapter XI. shows that this is Einstein's gravitational equation, provided we absorb as usual the electromagnetic energy-tensor into the matter tensor and identify the constant k with $(8\pi\kappa)^{-1}$.

CHAPTER XIII

SOLUTION OF EINSTEIN'S GRAVITATIONAL EQUATION.

AT this point it would appear advisable to turn aside for a while from the general question of the conformity of the laws of Nature with the principle of General Relativity, and deal briefly with the mathematical problem involved in solving the gravitational field equations proposed by Einstein. No general solution has been obtained so far, but a small number of particular solutions have been discovered. A brief summary of the calculations involved will be given in this chapter, but readers interested in the general literature will be well advised to consult the following original papers:

" Sitz. Preuss. Akad.," **32**, p. 688 (1916), by Einstein.

" Sitz. Preuss. Akad.," **7**, p. 189 (1916), by Schwarzschild.

" Sitz. Preuss. Akad.," **18**, p. 424 (1916), by Schwarzschild.

" Monthly Notices of the Roy. Ast. Soc.," **76**, p. 699, by de Sitter.

" Proc. Amsterdam. Acad.," **19**, pp. 197 and 447, by Droste.

" Ann. d. Physik.," **54**, p. 117 (1917), by Weyl.

" Phil. Mag.," **40**, p. 703 (1920), by Wilson.

" Phil. Mag.," **41**, p. 823 (1921), by Hill and Jeffery.

A very small acquaintance with the applications of Relativity to physical problems reveals the necessity for an altered attitude of mind towards the conception of co-ordinates. It is manifest in the fusion of space and time. It is brought home to us still more markedly at the outset of our immediate task. Consider, for a moment, the manner in which we have been accustomed to approach the solution of definite problems in gravitational mechanics, electrostatics or electromagnetic theory. We have, of course, to attempt the solution of a certain differential equation, Laplace's equation or the equation of propagation, subject to certain boundary conditions. We begin by " choosing a system of co-ordinates," they are Cartesian, polar, cylindrical, ellipsoidal, toroidal, or any other convenient curvilinear co-ordinates which happen to suit the

boundary conditions laid down. In each system the equation has a distinctive form which the mathematician recognises as peculiar to that system. If we obtain a formal solution for the potentials, scalar and vector, or for the intensities in terms of the independent variables, we experience no feeling of doubt or difficulty as to the physical interpretation of the solution; for we have at the beginning assigned a definite meaning to those variables in "choosing" the system of co-ordinates.

But matters are on quite a different footing if we propose to ourselves a similar problem in gravitational mechanics, taking Einstein's equation as our expression for the laws of gravitation.

Its outstanding property is its invariance, i.e., no matter what system of co-ordinates we "choose," the equation has no distinctive form which we recognise as peculiar to that particular system. The result is that if we succeed in obtaining a solution, i.e., in discovering functional forms for the g-potentials which satisfy the equation, we have no direct guide to a physical interpretation of the solution, for any so-called "choice of co-ordinates" would have been pointless as a preliminary to the purely mathematical problem involved in the solution. Such guidance must, in fact, be sought for from the *form of the solution itself*, and not from the form of equation. The most obvious clues will be found in the presence of certain singularities, in the absence of certain variables, and in the degenerate form of the solution when the approximations sufficient for Newtonian Dynamics are introduced.

Bearing these considerations in mind, let us proceed to obtain an exact solution of the equation "outside matter"

$$G_{\mu\nu} = 0$$

in the manner first indicated by Schwarzschild. It so happens that the clues mentioned above indicate that when physically interpreted it is the mathematical expression for the field of a single gravitating body. Of course, we can temper somewhat the rigorous relativist attitude of mind indicated above by a preliminary speculation concerning the form we might expect such a solution to take, if we adopted the system of co-ordinates which our acquaintance with Newtonian theory would suggest, viz., polar co-ordinates. At a great distance from the origin where we suppose the centre of the gravitating body to be situated, the element of separation between two events has a value whose square is given by

$$-\delta r^2 - r^2\delta\theta^2 - r^2 \sin^2 \theta\delta\phi^2 + \delta t^2, \quad . \quad . \quad (1)$$

where the usual significance is attached to the variables r, θ, ϕ, t.

It is plausible to assume that at a finite distance from O, the value of δs^2 would be given by an equation

$$\delta s^2 = -\chi_1 \delta r^2 - \chi_2 r^2 \delta\theta^2 - \chi_3 r^2 \sin^2\theta\delta\phi^2 + \chi_4 \delta t^2, \quad . \quad (2)$$

where χ_1, χ_2, χ_3, χ_4 are functions of the variables.

Analogy with the well-known solution in Newtonian theory and the consideration that the field is "stationary," i.e., not changing with lapse of time, would suggest that none of these functions would depend upon t, and that of the space co-ordinates, r alone would appear in their formal expressions. Furthermore, comparison with (1) would show that each of them would approach the value unity as r approaches infinity. And, finally, if the solution is to be symmetrical around the origin (a very natural hypothesis), and therefore independent of an orthogonal transformation of axes, the functions χ_2 and χ_3 should be identical.

As stated, these are but speculations as to possibilities. What we must now do is to discover if such a solution can be found.

Einstein's equations written fully are:

$$\partial\{\mu\nu, \alpha\}/\partial x_\alpha - \{\mu\alpha, \beta\}\{\nu\beta, \alpha\} + \{\mu\nu, \alpha\}\partial \log q/\partial x_\alpha \\ - \partial^2 \log q/\partial x_\mu \partial x_\nu = 0. \quad (3)$$

Now it is obvious that if we obtain one solution of this equation, we, at once, can obtain any number of solutions we please by transformation of co-ordinates. (This is not to be taken as saying we can obtain a *general* solution.) We can make use of this fact by imposing a certain simplifying restriction in the solution to begin with, which may be removed later by a transformation. Several such restrictions have been suggested by different investigators. The one suggested by Einstein, and also adopted by Schwarzschild, is

$$q = \sqrt{(-g)} = 1. \quad . \quad . \quad . \quad (4)$$

If we introduce this, (3) becomes

$$\partial\{\mu\nu, \alpha\}/\partial x_\alpha = \{\mu\alpha, \beta\}\{\nu\beta, \alpha\}. \quad . \quad . \quad (5)$$

In order to make the adoption of condition (4) as convenient as possible, Schwarzschild uses variables x_1, x_2, x_3, x_4, which are later interpreted as follows:—

$$x_1 = r^3/3, \quad x_2 = -\cos\theta,$$
$$x_3 = \phi, \quad x_4 = t.$$

The point of this interpretation is this. At a great distance the element of separation is from consideration of (1), given by

$$- (3x_1)^{-\frac{4}{3}}\delta x_1^2 - (3x_1)^{\frac{2}{3}}(1 - x_2^2)^{-1}\delta x_2^2$$
$$- (3x_1)^{\frac{2}{3}}(1 - x_2^2)\delta x_3^2 + \delta x_4^2. \quad . \quad . \quad . \quad . \quad (6)$$

In these variables $g_{11} = - (3x_1)^{-\frac{4}{3}}$, etc., and the condition (4) is satisfied at a great distance from the origin, which is not the case for the variables used in (1). At finite distances from O, the equation (2) transforms to

$$- f_1(x_1)\delta x_1^2 - f_2(x_1)(1 - x_2^2)^{-1}\delta x_2^2 - f_2(x_1)(1 - x_2^2)\delta x_3^2 + f_4(x_1)\delta x_4^2 \quad (7)$$

where (remembering that $\chi_2 = \chi_3$)

$$\left.\begin{array}{lll} f_1 \to (3x_1)^{-\frac{4}{3}} & \text{as } x_1 \to \infty \\ f_2 \to (3x_1)^{\frac{2}{3}} & ,, \quad ,, \\ f_4 \to 1 & ,, \quad ,, \end{array}\right\} \quad . \quad . \quad (8)$$

and

and the condition (4) implies that

$$f_1 f_2^2 f_4 = 1. \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Thus we have

$$g_{11} = -f_1, g_{22} = -f_2(1 - x_2^2)^{-1}, g_{33} = -f_2(1 - x_2^2), g_{44} = f_4$$
$$g_{\mu\nu} = 0 \text{ if } \mu \neq \nu.$$

Also, a simple calculation gives

$$g^{11} = f_2^2 f_4/g = -1/f_1 \text{ since } g = -1,$$
$$g^{22} = -(1 - x_2^2)/f_2, \; g^{33} = -1/f_2(1 - x_2^2), \; g^{44} = 1/f_4$$
$$g^{\mu\nu} = 0 \text{ if } \mu \neq \nu.$$

The next step is to calculate the Christoffel symbols. A considerable number of these are zero. Thus it is easy to show that $[\mu\nu, 1] = 0$, if $\mu \neq \nu$; the four surviving $[\mu\nu, 1]$ are:

$$[11, 1] = -\tfrac{1}{2}df_1/dx_1$$
$$[22, 1] = \tfrac{1}{2}df_2/dx_1 \,.\, (1 - x_2^2)^{-1}$$
$$[33, 1] = \tfrac{1}{2}df_2/dx_1 \,.\, (1 - x_2^2)$$
$$[44, 1] = \tfrac{1}{2}df_4/dx_1.$$

Of the $[\mu\nu, 2]$ symbols four are not zero. They are:

$$[12, 2] = -\tfrac{1}{2}df_2/dx_1 = [21, 2]$$
$$[22, 2] = -f_2 x_2(1 - x_2^2)^{-2}$$
$$[33, 2] = -f_2 x_2.$$

Only four of the $[\mu\nu, 3]$ and two of the $[\mu\nu, 4]$ survive, viz.:

$$[13, 3] = -\tfrac{1}{2}df_2/dx_1 \,.\, (1 - x_2^2)^{-2} = [31, 3]$$
$$[23, 3] = f_2 x_2 = [32, 3]$$
$$[14, 4] = \tfrac{1}{2}df_4/dx_1 = [41, 4].$$

It is now a straightforward matter to calculate the symbols of the second kind. They turn out to be as follows:—

$$\{11, 1\} = df_1/dx_1 \,.\, (2f_1)^{-1}$$
$$\{22, 1\} = -\, df_2/dx_1 \,.\, [2f_1(1 - x_2^2)]^{-1}$$
$$\{33, 1\} = -\, df_2/dx_1 \,.\, (1 - x_2^2)(2f_1)^{-1}$$
$$\{44, 1\} = df_4/dx_1 \,.\, (2f_1)^{-1}$$
$$\{12, 2\} = df_2/dx_1 \,.\, (2f_2)^{-1} = \{21, 2\}$$
$$\{22, 2\} = x_2(1 - x_2^2)^{-1}$$
$$\{33, 2\} = x_2(1 - x_2^2)$$
$$\{13, 3\} = df_2/dx_1 \,.\, (2f_2)^{-1} = \{31, 3\}$$
$$\{23, 3\} = -\, x_2(1 - x_2^2)^{-1} = \{32, 3\}$$
$$\{14, 4\} = df_4/dx_1 \,.\, (2f_4)^{-1} = \{41, 4\}.$$

The remaining $\{\mu\nu,\lambda\}$-symbols vanish.

Inserting these in equations (5), it will be found that if $\mu \neq \nu$ they are identically satisfied. So the equations become four in number,

$$\partial\{\mu\mu, \alpha\}/\partial x_\alpha = \{\mu\alpha, \beta\}\{\mu\beta, \alpha\},$$

where μ is not summed but put equal to 1, 2, 3, 4 in succession.

For convenience, write $\phi_1 = \log f_1$, etc., and we find after a few steps that these equations become

$$(\mu = 1)\ \ d^2\phi_1/dx_1^2 = \tfrac{1}{2}(d\phi_1/dx_1)^2 + (d\phi_2/\partial x_1)^2 + \tfrac{1}{2}(d\phi_4/dx_1)^2 \quad . \quad . \quad (10)$$

$$(\mu = 2)\ \ d(e^{\phi_2 - \phi_1} d\phi_2/dx_1)/dx_1 = 2 + e^{\phi_2 - \phi_1}(d\phi_2/dx_1)^2. \quad . \quad (11)$$

$(\mu = 3)$ This repeats equation (11).

$$(\mu = 4)\ \ d(e^{\phi_4 - \phi_1} d\phi_4/dx_1)/dx_1 = e^{\phi_4 - \phi_1}(d\phi_4/dx_1)^2. \quad . \quad . \quad . \quad (12)$$

Also from (9) we obtain

$$d\phi_1/dx_1 + 2d\phi_2/dx_1 + d\phi_4/dx_1 = 0. \quad . \quad . \quad (13)$$

From (12) we obtain

$$d^2\phi_4/dx_1^2 = d\phi_1/dx_1 \,.\, d\phi_4/dx_1 \quad . \quad . \quad (14)$$

so that

$$d\phi_4/dx_1 = ae^{\phi_1}$$

and

$$df_4/dx_1 = af_1f_4 \,. \quad . \quad . \quad . \quad . \quad (15)$$

where a is an integration constant.

Adding (10) and (14), we have

$$d^2(\phi_1 + \phi_4)/dx_1^2 = (d\phi_2/dx_1)^2 + \tfrac{1}{2}[d(\phi_1 + \phi_4)/dx_1]^2,$$

so that by means of (13) we obtain

$$2d^2\phi_2/dx_1^2 + 3(d\phi_2/dx_1)^2 = 0.$$

An integration of this yields

$$d\phi_2/dx_1 = 2/(3x_1 + b)$$

where b is another integration constant, so that

$$\phi_2 = \tfrac{2}{3}\log(3x_1 + b) + c,$$

or

$$f_2 = c(3x_1 + b)^{\frac{2}{3}},$$

c being a third integration constant.

The conditions (8) require c to be unity, so that

$$f_2 = (3x_1 + b)^{\frac{2}{3}}. \qquad (16)$$

Referring to (15), we see that by (9)

$$\begin{aligned} df_4/dx_1 &= a/f_2^2 \\ &= a/(3x_1 + b)^{\frac{4}{3}}, \end{aligned}$$

and thus

$$f_4 = 1 - a/(3x_1 + b)^{\frac{1}{3}} \qquad (17)$$

since $f_4 \to 1$ as $x \to \infty$.

From (9) it follows that

$$f_1 = 1/(3x_1 + b)[(3x_1 + b)^{\frac{1}{3}} - a]. \qquad (18)$$

We have not made use of equation (11) in obtaining these values for f_1, f_2, and f_4, but it is easy to show that it is satisfied by them.

To sum up, Schwarzschild's solution of Einstein's equation is:

$$\left.\begin{aligned} g_{11} &= -(3x_1 + b)^{-1}[(3x_1 + b)^{\frac{1}{3}} - a]^{-1} \\ g_{22} &= -(3x_1 + b)^{\frac{2}{3}}(1 - x_2^2)^{-1} \\ g_{33} &= -(3x_1 + b)^{\frac{2}{3}}(1 - x_2^2) \\ g_{44} &= 1 - a(3x_1 + b)^{-\frac{1}{3}} \end{aligned}\right\} \qquad (19)$$

Before proceeding, it may be as well once more to emphasise that this solution, as such, is the result of purely mathematical reasoning, and is in no way dependent on any physical interpretation of the variables. We are not even bound to assume that the variables do actually represent the co-ordinates or functions of co-ordinates, which we introduced above as an

assistance towards obtaining this solution. It is, however, only natural to make this assumption and test its conclusions, i.e., interpret x_1 as $r^3/3$, x_2 as $-\cos\theta$, x_3 as ϕ, and x_4 as t.

If we do so, and for convenience introduce the quantity R, which is equal to $(r^3 + b)^{\frac{1}{3}}$, we find that

$$\delta s^2 = (1 - a/R)\delta t^2 - \delta R^2/(1 - a/R) - R^2\delta\theta^2 - R^2 \sin^2\theta\delta\phi^2 \quad (20)$$

The astronomical consequences of this solution will be discussed in the second part of this chapter, where it will appear that comparison of observation with calculation is not so precise as to indicate whether in (20) R should be put equal to r or not.

This is but another aspect of that vagueness about the interpretation of variables which can only be cleared up in this case by sufficiently refined measures. True, if we care to make the problem a little more precise, and by adopting a procedure which is common in Newtonian dynamics, say that our gravitating body is a "particle," i.e., possesses no dimensions, we can remove some of this indecision about the meaning of our co-ordinates, for in such case we could naturally expect our solution to be finite and continuous everywhere except at the origin, $r = 0$. A glance at the second term on the right-hand side of (20) shows us that this is not so unless $a^3 = b$; for R is not equal to a, where $r = 0$, if $a^3 \neq b$. Without some such proviso, however, there is no obvious relation between a and b. The most natural physical interpretation for the constant a is that it is a measure of the "strength" of the gravitating body, and it will appear that it is equal to twice the Newtonian gravitating mass as far as any available measurement can tell.

But as for b, apart from the necessity of satisfying some extra condition such as that mentioned above, there is nothing to decide its value. It must be undoubtedly small compared, say, to the actual diameter of the sun. If we choose to put it equal to zero, we obtain

$$\delta s^2 = (1 - a/r)\delta t^2 - \delta r^2/(1 - a/r) - r^2\delta\theta^2 - r^2 \sin^2\theta\delta\phi^2, \quad (21)$$

the equation actually used in discussing the astronomical evidence. This, of course, is a solution which can only be considered as valid outside a sphere of radius a, and so would suggest a size for a gravitating "particle" closely connected with its mass, a conclusion which is not absurd, but is not verified by any known result.

But, of course, as stated above, there is really no limit to the number of forms which may be suggested for the element

of separation in the field of a single body. Thus, reverting for the moment to (2), viz.,

$$-\chi_1\delta r^2 - \chi_2(r^2\delta\theta^2 + r^2\sin^2\theta\delta\phi^2) + \chi_4\delta t^2$$

(using the condition $\chi_2 = \chi_3$), we see that Schwarzschild's solution satisfies the condition

$$\chi_1\chi_2^2\chi_4 = -1,$$

which is really Einstein's condition, $g = -1$.

Now Hill and Jeffery have recently shown that an exact solution can be obtained which satisfies the different condition

$$\chi_1 = \chi_2 = \chi_3.$$

The details of the solution can be worked out in a manner similar to that given above, and are to be found in the reference at the beginning of the chapter. But, as a matter of fact, Hill and Jeffery's solution can be obtained from Schwarzschild's by the transformation

$$r = r_1(1 + a/4r_1)^2,$$

b being considered as zero. On dropping the suffix after the transformation, we find that

$$\left.\begin{aligned}\chi_1 = \chi_2 = \chi_3 &= (1 + a/4r)^4\\ \chi_4 &= (1 - a/4r)^2/(1 + a/4r)^2\end{aligned}\right\} \quad . \quad . \quad (22)$$

It is not difficult to obtain forms of the solution suitable for Cartesian co-ordinates, for since

$$\delta r^2 + r^2\delta\theta^2 + r^2\sin^2\theta\delta\phi^2 = \delta x^2 + \delta y^2 + \delta z^2$$

and

$$(r\delta r)^2 = (x\delta x + y\delta y + z\delta z)^2,$$

the solution (21) can be written

$$(1 - a/r)\delta t^2 - (\delta x^2 + \delta y^2 + \delta z^2) \\ - a(x\delta x + y\delta y + z\delta z)^2/r^3(1 - a/r).$$

This gives a scheme of g-potentials which can be easily written down, e.g.,

$$g_{11} = -(1 + a'x^2/r^3),\ g_{12} = -a'xy/r^3, \text{ etc.},$$

where $a' = a/(1 - a/r)$.

Hill and Jeffery's solution can be similarly transformed into the rather elegant Cartesian form

$$(1 - a/4r)^2(1 + a/4r)^{-2}\delta t^2 - (1 + a/4r)^4(\delta x^2 + \delta y^2 + \delta z^2).$$

The latter also gives a very useful approximate solution, viz.,

$$(1 - 2\phi)\delta t^2 - (1 + 2\phi)(\delta x^2 + \delta y^2 + \delta z^2),$$

where ϕ is the usual Newtonian potential function. As a matter of fact, Eddington makes a very interesting and ingenious use of this form in order to obtain Einstein's equation inside matter. The details are in Chapter VI. of Eddington's "Report," and the reader will find it instructive to compare this method with Einstein's own, which was given in Chapter XI. above.

For further details, and especially for solutions of the equations at a point within matter, the reader should consult the references given earlier.

Astronomical Consequences of Einstein's Equation.

Equations of Motion of a Particle in the Field of a single Gravitating Body.

It will be rather more convenient to use the "Lagrangian" form of the equations of motion, viz.,

$$d(\partial L/\partial \dot{x}_\mu)/ds - \partial L/\partial x_\mu = 0 \quad . \quad . \quad (23)$$

where

$$2L = \gamma \dot{t}^2 - \gamma^{-1}\dot{r}^2 - r^2\dot{\theta}^2 - r^2 \sin^2\theta\dot{\phi}^2$$

and we employ the more usual symbols instead of x_1, x_2, x_3, x_4 and write γ for $1 - k/r$, k being a constant.

The fourth of the equations (23) is

$$d(\gamma\dot{t})/ds = 0.$$

Hence $$\gamma\dot{t} = \text{a constant} = b,$$

or $$b\delta s = (1 - k/r)\delta t. \quad . \quad . \quad . \quad (24)$$

The third of the equations is

$$d(\partial L/\partial\dot{\phi})/ds - \partial L/\partial\phi = 0,$$

i.e., $$d(r^2 \sin^2\theta\dot{\phi})/ds = 0$$

so that $$r^2 \sin^2\theta\dot{\phi} = \text{a constant}.$$

We can, without loss of generality, take this constant to be zero, and put

$$d\phi/ds = 0,$$

or $$\phi = \text{a constant}.$$

This means that the motion is in one plane.

Turning to the θ co-ordinate we have

$$d(\partial L/\partial\dot{\theta})/ds - \partial L/\partial\theta = 0.$$

Now $$\partial L/\partial\theta = -r^2 \sin\theta \cos\theta\dot{\phi}^2$$
$$= 0.$$

Therefore $$r^2\dot{\theta} = \text{a constant},$$

or $$r^2 d\theta/ds = h. \qquad (25)$$

This result is the analogue of the law of equal areas in Newtonian theory.

There is no necessity to trouble about the remaining equation; (24) and (25) combined with the form for δs^2 are all we require to obtain the differential equation of the orbit.

By (24)

$$dt/ds = b/\gamma.$$

Hence, using the form of δs^2, we obtain

$$\gamma(b/\gamma)^2 - \gamma^{-1}(dr/ds)^2 - r^2(d\theta/ds)^2 = 1.$$

Using (25) we have

$$(dr/ds)^2 + \gamma h^2/r^2 = b^2 - \gamma.$$

But (25) also shows that

$$dr/ds = h/r^2 \,.\, dr/d\theta.$$

Hence

$$h^2/r^4 \,.\, (dr/d\theta)^2 + \gamma h^2/r^2 = b^2 - \gamma.$$

As is usual in the mathematical analysis of motion "under central forces," we introduce a symbol for the reciprocal of the radius vector, $1/r$. Call it u. Then

$$(du/d\theta)^2 + (1 - ku)u^2 = uk/h^2 + (b^2 - 1)/h^2.$$

This gives a differential equation for the orbit in terms of plane polar co-ordinates; but a further differentiation with respect to θ will give us a differential equation differing by one term from the familiar second order equation for motion round a centre attracting as the inverse square Differentiating in this way, and cancelling out $du/d\theta$ from each term, we obtain

$$d^2u/d\theta^2 + u - \tfrac{3}{2}ku^2 = k/2h^2. \qquad (26)$$

As mentioned, we are accustomed in Newtonian theory to the equation for an orbit in the form

$$d^2u/d\theta^2 + u = M/h^2 \qquad (27)$$

where M is the gravitational mass of the attracting body. As Newtonian theory is such a good approximation, we can avail ourselves of this to identify the constant k in Einstein's orbit equation as $2M$, so that numerical tests become possible. So we write (26) as

$$d^2u/d\theta^2 + u - 3Mu^2 = M/h^2. \quad . \quad . \quad (26\text{A})$$

The exact solution of (26A) involves elliptic functions, but on account of the smallness of the third term in comparison with the others, in the case of any of the planetary orbits in the solar system, an approximate solution in trigonometrical functions sufficiently accurate for immediate purposes can be obtained.* Before working it out, we shall exhibit the orders of magnitude involved.

The earth's mass is about 5·5 times the mass of an equal globe of water (6400 kilometres in radius), i.e., about 6×10^{21} metric tons, or $\frac{1}{9}$ of the Relativity astronomical unit of mass.† The sun's mass is about one-third of a million times the earth's, i.e., about $1{\cdot}5 \times 10^5$ Relativity units, or $M = 1{\cdot}5 \times 10^5$, approximately. The average distance of Mercury, the innermost planet, from the sun is about 60,000,000 kilometres, or 6×10^{12} cms. So for the orbit of Mercury Mu (i.e., M/r is about $2{\cdot}5 \times 10^{-8}$). In Newtonian theory h^2/M is the latus rectum of the ellipse obtained from equation (27), and so h^2/M is of the same order of magnitude as the radius vector r. Consequently the extra term in (26A), $3Mu^2$, is about 10^{-7} of any of the terms in the Newtonian equation, and so in a solution we can neglect terms involving the square of this ratio.

Writing (26A) as

$$d^2u/d\theta^2 + u = \lambda + 3Mu^2,$$

where $\lambda = M/h^2$ and is the reciprocal of a length, we see that if we neglect the term $3Mu^2$, we obtain as an approximate solution

$$u = \lambda(1 + e\cos\theta),$$

* For an exact solution, see a paper by Forsyth ("Proc. R.S.," A. 682 (1920)).

† In Newtonian theory $2M/r$ is the square of the velocity "from infinity," and so units of M must be chosen to suit our unit of velocity, which is c cms. per sec. Consequently, r being in cms., M is not expressed in the usual astronomical unit of gravitational mass (that which concentrated at a point attracts another unit with force 1 dyne at a distance 1 cm., about 15×10^6 grams or 15 metric tons), but in an enormously greater unit, viz., c^2 times the usual astronomical unit or $1{\cdot}35 \times 10^{28}$ grams, or $1{\cdot}35 \times 10^{22}$ metric tons.

which is the polar equation of a conic with λ as the reciprocal of its latus-rectum and e as its eccentricity, the initial line being so chosen as to pass through the perihelion of the planet, where r has its minimum value and u its maximum.

To proceed to a second approximation, we put this value in the term $3Mu^2$ of (26A), and obtain as the differential equation

$$\begin{aligned} d^2u/d\theta^2 + u &= \lambda + 3M\lambda^2 + 6M\lambda^2 e\cos\theta + 3M\lambda^2 e^2\cos^2\theta \\ &= \lambda + \mu + \nu\cos\theta + \kappa\cos 2\theta, \end{aligned}$$

where

$$\begin{aligned} \mu &= 3M(1 + \tfrac{1}{2}e^2)\lambda^2 \\ \nu &= 6Me\lambda^2 \\ \kappa &= \tfrac{3}{2}Me^2\lambda^2. \end{aligned}$$

μ, ν, κ are obviously of the order $10^{-7}\,\lambda$ at most, ν and κ being even smaller if the eccentricity of the approximate ellipse is small, as is the case for nearly all the planets in the solar system.

This is a well-known type of differential equation, and its solution is

$$u = (\lambda + \mu)(1 + e\cos\theta) + \tfrac{1}{2}\nu\theta\sin\theta - \tfrac{1}{3}\kappa\cos 2\theta \quad . \quad (28)$$

as can be verified by trial. Of course, we could write any arbitrary constant where we have written e, so far as satisfying the differential equation is concerned. Naturally, however, the solution must degenerate into the Newtonian ellipse if we neglect μ, ν, and κ.

Equation (28) represents an orbit slightly perturbed from the ellipse

$$u = (\lambda + \mu)(1 + e\cos\theta)$$

by amounts depending on the remaining two terms on the right-hand side. Of these, the term $\frac{1}{2}\nu\theta\sin\theta$ produces ultimately the greater effect, for it may, during the course of the $(n+1)^{\text{th}}$ revolution, after the initial instant, acquire a value $\pm\, n\nu\pi$, which increases with n, while the other term never exceeds $\frac{1}{3}\kappa$ numerically.

The interesting point is to determine where the perihelia occur. In the case of the ellipse, they occur at $\theta = 0, 2\pi, 4\pi$, etc. ($\pi, 3\pi, 5\pi$, etc., corresponding to aphelia). To obtain their angular position in the perturbed orbit, we must choose the alternate solutions of

$$du/d\theta = 0,$$

i.e.,

$$\{(\lambda + \mu)e - \tfrac{1}{2}\nu\}\sin\theta - \tfrac{2}{3}\kappa\sin 2\theta = \tfrac{1}{2}\nu\theta\cos\theta.$$

There will be a solution near $\theta = 2n\pi$; so putting $\theta = 2n\pi + \alpha$, we obtain on writing α for sin α, and unity for cos α,

$$\{(\lambda + \mu)e - \nu - 4\kappa/3\}\alpha = n\nu\pi,$$

or practically

$$\left.\begin{aligned} \alpha/n &= \pi\nu/\lambda e \\ &= 6M\lambda\pi \end{aligned}\right\} \qquad (29)$$

Equation (29) gives an advance in the perihelion per revolution of the amount $6M\lambda\pi$ radians. Now $M = h^2\lambda$, and λ is the reciprocal of the latus-rectum of the approximate ellipse, which is equal to $b(1 - e^2)^{\frac{1}{2}}$, where b is the minor semi-axis. Also, referring back, h is practically twice the area swept out per unit time by the radius vector, i.e., $h = 2\pi ab/T$ where a is the major semi-axis and T is the period of a revolution (in $1/c$ second as unit).

Hence the advance of the perihelion per revolution is

$$\frac{24\pi^3a^2}{T^2(1 - e^2)} \text{ radian.} \qquad (30)$$

This result has received a most remarkable confirmation in the case of the planet Mercury. Its value when multiplied by the number of revolutions in a century is 43 seconds of arc. By Newtonian theory the perturbing action of the remaining planets should cause the perihelion of Mercury to advance 532″ per century. The mass of any one of these planets is so small compared to that of the sun, that the treatment of its gravitational field at Mercury by the Einstein method would yield results only differing from those obtained by the earlier method by amounts much too minute to be perceptible. Hence Einstein theory would still give 532″ as the perturbing action of the other planets. But the observed value is 574″ per century. It was Le Verrier who first pointed out the discrepancy, and no explanation entirely acceptable to astronomers seems to have been forthcoming, until Einstein pointed out that the difference was accounted for by his treatment of the gravitational field of the sun. By its action alone on his view there is an advance of 43″, to which is added, of course, to the 532″ due to the other planets, and close agreement with observation is thus obtained. The arresting feature of the result is the absence of any adjustable constants in the formula. The values of a, T, e are well-known astronomical data. Of course, there must be similar advances in the case of the other planets, but they are smaller, being about 8″ per century for

Venus, about 4″ per century for the Earth, and still smaller values for the remaining planets. Apparently, it is next to impossible to test these results. It seems that astronomers do not directly observe the advance, but the product of the advance and the eccentricity. This product is about 8″ in the case of Mercury, whose eccentricity is ·2, but the eccentricities of Venus and the Earth are so small that the products in their case are only about one-twentieth of a second.

It will be as well to give a little further thought to this result before proceeding. In (28) we have the polar equation of an orbit. Interpreted in the usual way r represents the distance from the sun, θ the angle from a certain initial line. We might consider this orbit drawn absolutely correctly to scale on a large sheet of paper. A series of adjacent points $P_1, P_2, P_3, \ldots, P_m$ could be marked on the curve; each one would have an appropriate r and θ, and by means of (24) and (25) each one could have an appropriate t marked beside it. Yet we on the Earth should not actually obtain $t_2 - t_1$ as the time of passage of the planet between the positions represented by P_1 and P_2 on our plans; we should obtain $\gamma^{\frac{1}{2}}(t_2 - t_1)$. Neither should we get $r_2 - r_1$, as the radial displacement; we should get $(r_2 - r_1)\gamma^{-\frac{1}{2}}$. Our measure of the displacement transverse to the radius vector would agree with the plan.

This is so, because if $\delta t'$, $\delta r'$, $r'\delta\theta'$ are our measures made in our own terrestrial frame, we must write

$$\delta s^2 = \delta t'^2 - \delta r'^2 - r'^2\delta\theta'^2$$

(ignoring the ϕ co-ordinate as of no importance in a plane orbit), because our axes are Galilean and our instruments are fixed in this frame. So to obtain the form for δs^2 we must put

$$\begin{aligned} \delta t' &= \gamma^{\frac{1}{2}}\delta t \\ \delta r' &= \gamma^{-\frac{1}{2}}\delta r \\ r'\delta\theta' &= r\delta\theta. \end{aligned}$$

This reveals a lack of agreement between our measures and the plan. There are two ways of describing this discrepancy. One is to say that t is the time which would be given by a clock at rest in the solar gravitational field, and not accelerated like ours, and r and θ are measures obtained by rulers, protractors, telescopes, etc., also at rest in the gravitational field. Compared to such a clock our actual clocks go slow, because, since $\gamma < 1$, they give a smaller number than δt for their measure of a certain interval of time; compared to such a ruler, our actual ruler, if held parallel to the solar field, is short,

since it gives a larger number than δr for a certain radial measure; if held transverse to the field it agrees. This way of looking at the matter appeals undoubtedly to our feeling that we must have something tangible for gravitation to exert an effect on. It no longer is a matter of exerting force on a body, but the field exerts an influence on our metrical relations with bodies in the field, which we phrase most easily for ourselves by saying that it makes our clocks go slow and shortens our rulers if held parallel to the field. But, of course, such running slow and shortening has nothing absolute about it—it must be purely relative to clocks and rulers disposed in a highly unnatural manner, and assumed to be free from what is otherwise a universal behaviour. In this connection it is very essential to bear in mind that it is the Newtonian potential and not intensity of gravitational force which determines the effect on our measures.

The other way of looking at the matter is one which cannot be explained without some preliminary discussion on the possibility of our space-time continuum being curved or "warped." But if, instead of drawing the curve given by equation (28) on a *plane* sheet of paper, we drew it on a curved surface, the lengths of geodesic lines from the focus to the curve being represented by r, we might by properly choosing the surface make our measures of length fit with the dimensions of the plan; but the difficulty about the measured time would still remain. We shall return to this matter later.

Let us once more take up the analysis and consider the velocity of propagation of light in the field. This is determined by the condition $\delta s = 0$, i.e.,

$$\gamma\delta t^2 - \gamma^{-1}\delta r^2 - r^2\delta\theta^2 = 0 \quad . \quad . \quad . \quad (31)$$

in a given plane.

Hence if v be the velocity of light as determined by the co-ordinates r, θ, t,

$$v^2(\cos^2\psi + \gamma\sin^2\psi) = \gamma^2 \quad . \quad . \quad . \quad (32)$$

where ψ is the angle between the direction of the light at a point and the radius vector from the gravitating centre to the point.

Along the radius vector this gives

$$v = \gamma = 1 - 2M/r.$$

At right angles to r it gives

$$\begin{aligned} v &= \gamma^{\frac{1}{2}} \\ &= 1 - M/r \text{ approximately.} \end{aligned}$$

This gives the velocity as different in different directions; but, of course, this velocity is not the actual measure made by observers, who are naturally always in a Galilean frame. We should again have to postulate observers unnaturally fixed with their instruments in the field. Actual observers get $\delta t'(=\gamma^{\frac{1}{2}}\delta t)$, and not δt for the measured time of the light between two points, and $r'\delta\theta'(=r\delta\theta)$ and $\delta r'(=\gamma^{-\frac{1}{2}}\delta r)$ as the components of the distance between the points, so that for them the speed of light is

$$(\delta r'^2 + r'^2\delta\theta'^2)^{\frac{1}{2}}/\delta t',$$

which is unity. Yet in dealing with celestial phenomena, such as planetary motion, we refer an orbit to the sun as centre, and so we choose axes with the sun as origin, and referred to these axes the path is no longer the ideal ellipse with planetary perturbations superposed. In a similar manner, if we choose the sun as origin, the path of a ray of light through the solar system is no longer straight if we derive the co-ordinates of points on it from measurements made by us on the earth, using Einstein's law of gravitation and the values of the potentials inferred from it. For since v is variable, the wave from it must slew round. Let us calculate the bending for a ray of light which passes a gravitating centre at a nearest distance a.

Choose axes so that OY passes through A, the nearest point of the ray to O, where the centre is situated, and so that OX is parallel to the direction of the ray at this point. Choose two consecutive points P and P′ on the ray, and let PQ and P′Q′ be elements of normals to the ray, so that Q and Q′ are consecutive points on an adjacent ray. The angle between PQ and P′Q′ is the amount of deviation of the ray between P and P′. This angle is

$$(\mathrm{QQ}' - \mathrm{PP}')/\mathrm{PQ}.$$

If $v + \delta v$ is the velocity of Q and $\mathrm{PP}' = \delta s$, so that the time between P and P′ or Q and Q′ is $\delta s/v$, this angle is equal to

$$\delta v\delta s/(v\delta n)$$

where

$$\delta n = \mathrm{PQ}.$$

Hence the amount of bending of the ray is equal to the integral

$$\int v^{-1}dv/dn \,.\, ds \quad . \quad . \quad . \quad . \quad (33)$$

between assigned limits.

Now by (32)

$$v(1 - 2M \sin^2 \psi/r)^{\frac{1}{2}} = 1 - 2M/r,$$

or
$$\left.\begin{aligned} v &= 1 - 2M/r + M \sin^2 \psi/r + \text{higher powers of } M/r \\ &= 1 - M(1 + \cos^2 \psi)/r + \text{etc.} \end{aligned}\right\} \quad (34)$$

Combining (33) and (34) will give the required result. In the case of a ray passing near to the sun, $M = 1{\cdot}5 \times 10^5$, and a (taken as the sun's radius) is approximately 8×10^9, so that the maximum value of M/r is about 2×10^{-5}. As $v^{-1}\, dv/dn$ is of the order M/r^2, it is clear that all but a negligible part of the bending is contributed by the part of the ray lying between two points equidistant from A and situated at a distance, say $100a$, from it (roughly half the distance of the earth from the sun). The bending is so small that the ordinate y of any point on this portion of the ray may be assumed to be a without practical error; also ds may be replaced by dx and d/dn by $\partial/\partial y$ in (33). Hence the total bending of a ray coming from a great distance, passing the sun and travelling to a great distance, is equal to

$$\begin{aligned} &2\int_0^\infty v^{-1}\partial v/\partial y \,.\, dx \\ &= 2\int_0^\infty \partial[M(1 + \cos^2 \psi)/r]/\partial y \,.\, dx \\ &= 2\int_0^\infty \partial(M/r + Mx^2/r^3)/\partial y \,.\, dx \\ &= 2M\int_\infty^0 (a/r^3 + 3ax^2/r^5)dx, \end{aligned}$$

since $\partial r/\partial y = y/r = a/r$.

Putting

$$\begin{aligned} x &= a \tan \theta \\ r &= a \sec \theta \\ dx &= a \sec^2 \theta d\theta, \end{aligned}$$

we see that the bending is equal to

$$\begin{aligned} &2M/a \,.\int_0^{\pi/2} (\cos \theta + 3 \sin^2 \theta \cos \theta)d\theta \\ &= 4M/a. \end{aligned}$$

It is interesting to notice that a bending of light would be quite conceivable on Newton's corpuscular theory, and, in fact, speculation on such a possibility did engage his attention. Were the hypothetical corpuscles subject to the gravitation of

stellar bodies according to his law, they would travel along a branch of a hyperbola of very great eccentricity in passing such a body, the speed of the corpuscle *increasing* as it approached the gravitating mass, the very reverse of Einstein's conclusion. Further, it is easy to show that in this case the bending would be $2M/a$, half of that predicted by Einstein. For example, the bending would be the acute angle between the asymptotes of the hyperbola. Now on Newtonian theory

$$\begin{aligned} v^2 &= 2M/r + \text{the square of the velocity at infinity} \\ &= 2M/r + 1. \end{aligned}$$

But v^2 is also $2M/r + M/R$ where R is the major semi-axis of the hyperbola.

Hence $\qquad R = M.$

But $\qquad a = (e - 1)R,$

where e is the eccentricity.

Therefore $\qquad e = 1 + a/M$
$\qquad\qquad = a/M$ practically.

Since the acute angle between the asymptotes is $2(e^2 - 1)^{-\frac{1}{2}}$, or $2/e$ practically, we obtain for the bending $2M/a$.

It can also be shown that, using the electromagnetic theory and assuming that radiation has a mass subject to gravitational action, the result $2M/a$ is again obtained. (See Eddington's "Report," Chapter V., pp. 55, 56.)

In the case of the sun, $4M/a$ radian is equal to 1·74 seconds of an arc, and it is to this value, and not the half of it, that the results of the British Eclipse Expeditions of 1919 give great support.*

A third crucial test of Einstein's work on gravitation has exercised a good deal of careful attention both on the theoretical and experimental sides. It is claimed that the theory predicts a shift of the Fraunhofer lines in the solar spectrum towards the red by amounts equal to those which would be given by a radial velocity of recession of 0·63 kilometre per second. Now not only is the experimental evidence on this matter indecisive at the moment, but some physicists such as Larmor, Jeans, and Cunningham have expressed doubt as to the validity of the deductions of the result from Einstein's theory. The deduction itself begins by considering some periodic mechanism situated at rest in axes attached to a gravitating body. Referred to Galilean axes in motion at the locality of the mechanism with the acceleration characteristic

* Completely confirmed by the American Expedition investigating the eclipse of September, 1922.

of the place, its period is δt, let us say, and the mechanism has a displacement δx, δy, δz in these axes during the time δt. Then, as we have seen,

$$\delta t^2 - \delta x^2 - \delta y^2 - \delta z^2 = g_{11}\delta x_1^2 + \ldots + 2g_{34}\delta x_3\delta x_4$$

where δx_4 is the period of the mechanism in the gravitational frame, and δx_1, δx_2, δx_3 its displacement in the time δx_4. If the mechanism is at rest in the latter axes, $\delta x_1 = 0 = \delta x_2 = \delta x_3$ and

$$\delta t^2 - \delta x^2 - \delta y^2 - \delta z^2 = g_{44}\delta x_4^2.$$

The left-hand side of this equation is $\delta t^2(1 - v^2)$, and its square root is the proper time of the period, i.e., the period as determined when the mechanism is at rest in a frame free from gravitation. Now suppose that an exactly similar mechanism is situated at rest at another place in the gravitational field x_1', x_2', x_3', then we can prove that the proper time is also equal to the square root of

$$g_{44}'\delta x_4'^2$$

where $\delta x_4'$ is the period of the mechanism in this new position.* Hence

$$g_{44}\delta x_4^2 = g_{44}'\delta x_4'^2$$

and so the periods of the mechanisms would differ if situated in different parts of a gravitational field, although they would agree if they were brought together.

In making a test of this conclusion, it is assumed that the radiating mechanisms in two atoms of the same element, sodium, for example, or cyanogen or nitrogen, are such mechanisms as we have referred to, i.e., that for all atoms of one kind the interval δs for a given spectral line is the invariant thing which remains unaltered by displacement from one place to another. It is this assumption which has had to endure a certain amount of adverse criticism. Choosing axes fixed in the sun, and considering the first atom as fixed on the surface of the sun, and the second on the earth, we have, making use of the expression for δs^2,

$$(1 - 2M/r_1)\delta t_1^2 = (1 - 2M/r_2)\delta t_2^2 - (1 - 2M/r_2)^{-1}\delta r_2^2 - r_2^2\delta\theta_2^2$$

where δr_2, $r_2\delta\theta_2$ is the earth's displacement in the time δt_2.

* The reader is warned that the accented symbols refer to another event at a different place in the same axes, and not the same event referred to another system of axes.

In this equation, r_1 being the sun's radius and r_2 that of the earth's orbit, it appears that $2M/r_2$ is negligible compared with $2M/r_1$; further, $\delta r_2/\delta t_2$ and $r_2\delta\theta_2/\delta t_2$ being the components of the earth's velocity, their squares are also of the order M/r_2, and so are negligible. Practically, we have

$$\begin{aligned} \delta t_2 &= (1 - 2M/r_1)^{\frac{1}{2}}\delta t_1 \\ &= (1 - M/r_1)\delta t_1. \end{aligned}$$

That is, the period of the terrestrial atom is a little less than that of the solar, and so the solar lines should show a displacement to the red as compared with the terrestrial. The quotient of the solar mass by its radius (in Relativity units) is about 2×10^{-6}. In that part of the spectrum in which investigation has been made, the displacement should be about ·008 Angstrom units. As solar lines are subjected to many disturbing influences such as the pressure-shift, the Döppler effect due to ascending and descending masses of gas for which it is difficult to make allowance, the practical testing of this conclusion is beset with every serious difficulty. In order to eliminate spectral displacements arising from differences of pressure between the sun's reversing layer and the terrestrial arc, special attention has been paid to the lines in the cyanogen band at $\lambda = 3883$, which are known to be free from the pressure effect. Further, since ascending and descending currents of gas would cause no first order Döppler effect at the sun's limb, observations have been made at the sun's limb as well as at the centre of the disc, and the Döppler effect due to the sun's rotation has been avoided by employing the parts of the limb near the sun's poles. But there still remain unsettled points such as the weight to be ascribed to results derived from weak lines, and as compared with those derived from strong, and the question of a displacement of the cyanogen lines towards the violet due to the proximity of metal lines, a matter which has received some attention in the researches of Grebe and Bachem. While the results of St. John in America in 1917 are distinctly unfavourable, those of Evershed in 1918 and Grebe and Bachem in 1919 offer considerable support. A mean of all the known experimental results, which have sufficient weight, gives ·004 or half the calculated value.

The position is certainly not satisfactory, as the deduction is certainly not convincing to some minds, and the experimental evidence gives no decided lead one way or the other. It should be stated that Einstein himself maintains that his whole theory of gravitation stands or falls by the success or

failure of this test; on the other hand, some physicists like Jeans maintain that if we do not insist on the special interpretation of *ds* made by Einstein (which we will meet later when treating of the four-dimensional space-time continuum), the theory can still hold, and embrace the extended planetary motion and the curvature of light; it will become "less rich and less inclusive, but at the same time less beset with difficulties." Those interested will find this matter dealt with very fully in a discussion on Relativity held by the Royal Society, and printed in the "Proceedings," Vol. A 97, No. 681, March, 1920.

The pages of "Nature" during the early months of 1920 also contain some interesting articles and letters in which this question, among others, receives a good deal of attention. In this connection it should be mentioned that in the same volume of this journal alternative explanations of the Einstein curvature of the light ray are put forward, e.g., the possibility of a refracting atmosphere round the sun, or, as suggested by Professor Anderson of Galway, a refraction due to the passage of the light on its way to the camera into the (possibly) cooled column of air in the eclipse shadow. Both these suggestions are adversely criticised (and in the writer's judgment successfully disposed of) by Crommelin and Schuster.

The matter of the spectral displacement is so important that an alternative, and somewhat more elementary way, of stating the proof may not be out of place, which avoids bringing in the symbolism of the proper time and exhibits rather more clearly the special parts played by the general principle of Relativity and the principle of Equivalence as distinct from one another.

Suppose we have two identical sources of light at points A_1 and A_2 on the axis OX of a frame of reference free from gravitation. An observer at O obtains the same period T for each. If we now consider an observer situated in a frame of reference moving uniformly with respect to the former with a velocity u parallel to OX, he would also observe that the periods are equal. They are, of course, no longer T, but $T/\alpha(1 + u)$ by Chapter II.; but they are still equal to one another, and this equality is independent of the relative velocity of the frames. If General Relativity is true, the equality should still persist even if the second frame of reference were accelerated along OX. No doubt the periods would be varying according to the Döppler effect, but the observer would perceive equal frequencies at one instant. In fact, in the first frame, the front of a train of n waves from A_1 arrives at O simul-

taneously with the front of a train of waves from A_2, and the end of the first train also arrives simultaneously with the end of the second. But the coincidence of two events at the same time and place in one frame remains a coincidence in any other frame. So these two synchronisms remain as synchronisms in any other frame; no doubt the interval involved between the arrival of the front and back of each train will alter with the frame employed, but in a given frame it will be the same for each. Thus far we are employing nothing but the principle of Relativity. Now comes the principle of Equivalence. In the accelerated frame the sources are moving with the same acceleration parallel to OX, either to or from the observer at O, and maintaining a constant distance between them (i.e., relative to an observer fixed on either). Hence, if Equivalence is true, these two sources falling in a field of gravitation will emit light beams which will have the same frequency as received by an observer fixed in the field. The frequencies will no doubt be increasing as time goes on, on account of increasing speed of the sources towards observer, but at any one instant they are equal. Let T now stand for the common value of this period as received by the observer at a given instant when the sources are at $OA_1 = x_1$ and $OA_2 = x_2$ respectively, and their common speed is v. But the actual period of A_1 is not calculated by Döppler's principle to be $T(1 + v)$, but $T(1 + v - gx_1)$; for the light now reaching O was emitted at an instant x_1 earlier (the speed of light is taken as unity, and we are omitting quantities of the second order), and the velocity of A_1 then was $v - gx_1$, so the true period of the first source, i.e., the period of the light emitted by it *if it were fixed at* A_1 would be

$$T_1 = T(1 + v - gx_1).$$

Similarly, the true period of the second source *if fixed at* A_2 is

$$T_2 = T(1 + v - gx_2).$$

Hence, as v, gx_1, gx_2 are of the first order,

$$T_2 = T_1(1 - gl),$$

where $l = x_2 - x_1$; or the period of the source at the lower potential (A_1) is longer than that at the higher potential. It is noteworthy that while we have ignored certain factors of the second order in v, etc., the ratio of the two periods involves unity minus a Newtonian potential difference, in analogy with

the factor $1 - M/r$ arrived at in the fuller treatment of the solar gravitational field. In fact, as already stated, it is the difference in potential between the two places, as distinct from intensity of gravitational field, which determines the difference in period of sources fixed in these places. The field might be very weak at and between both places, but provided there is a large enough difference of potential, the difference in period would be marked. When put in this way, there seems to be little hope, as Eddington says in his " Report " (p. 58), " of evading the conclusion that a displacement of the Fraunhöfer lines is a necessary and fundamental condition for the acceptance of Einstein's theory ; and that if it is really non-existent, under conditions which strictly accord with those here postulated, we should have to reject the whole theory *constructed on the principle of Equivalence.** Possibly a compromise might be effected by supposing that gravitation is an attribute only of matter in bulk, and not of individual atoms ; but this would involve a fundamental re-statement of the whole theory."

In the discussion referred to above, Cunningham suggested that the failure to detect the spectral shift, if confirmed, would involve some difference between a simulated gravitational field such as can be entirely transformed away, and a real gravitational field which is associated with the presence of matter and cannot be transformed away in its entirety, but only locally. Such a difference would be of such a nature as not to interfere with the similarity of the real and simulated fields as regards motion of matter or transmission of light. It is certainly noteworthy that the difficulty has arisen in connection with emission of light, i.e., atomic processes, and as Eddington points out, if the displacement of the solar lines be confirmed, it will be the first experimental evidence that Relativity holds for quantum phenomena.

As regards other solutions of the equation for a single body, one advantage of Hill and Jeffery's result is that it makes the velocity of light at a point independent of the direction of the ray, the value being

$$(1 - a/4r)/(1 + a/4r)^3. \quad . \quad . \quad . \quad (35)$$

But it must be carefully remembered that this new expression of the separation element leads to no contradiction of the known physical measurements, which, be it noted, all concern

* The italics are mine We could still, for all that, retain the principle of Relativity in its most general form.

the angle θ and not the co-ordinate r, about which there is a certain indeterminateness in interpretation. Thus (22), leading to (35), still gives $4M/a$ for the deviation of a light ray to the order M/a, which is all that can be observed. For the path of a particle in the field, (22) gives the equation

$$d^2u/d\theta^2 + u - 6M^2u/h^2 = M/h^2,$$

which, though different to the equation (26A), derived from (21), nevertheless yields the same result, $6M^2\pi/h^2$, for the advance of the apse line per revolution. The reader, who is interested to pursue this matter a little further, will find some instructive and critical remarks by M. Painlevé in the "Comptes Rendus" (24th October and 14th November, 1921) This writer explores the possibilities of considering a still more general form for δs^2, such as

$$(1 - k/f(r))\delta t^2 - f^2(r)(\delta\theta^2 + \sin^2\theta\delta\phi^2) - f'^2(r)(1 - k/f(r))^{-1}\delta r^2,$$

where r is interpreted as the radius vector and $f(r)$ is an arbitrary function which approaches the value r, as r grows larger. He points out that in the actual state of our measures, nothing decisive can be established from a consideration of trajectories alone. Only very precise observations of times, such as the durations of the years of each planet, will yield any data which can have a decisive influence on our choice from the alternative expressions for δs^2; and even then the planets will have to be without satellites if we are to avoid the formidable mathematical difficulties raised by the problem of three bodies. Some remarks by Dr. Dorothy Wrinch, bearing on the question generally as to the correct application of the scientific method in the dilemma occasioned by such criticisms as those of Painlevé, will also be found very helpful by the reader with a philosophical mind. They appear in "Nature," 23rd March, 1922.

Propagation of Gravitational Waves.

The field of a single gravitating centre, as expressed in any of the solutions obtained above, is a static field from the point of view of an observer in the same frame of reference as the centre. The field of a number of centres, such as a planetary system, will alter with time, i.e. the solution for the g-potentials will involve x_4 as well as x_1, x_2, and x_3. This is, of course, also true in Newtonian theory. But whereas in the latter case the conception of an absolute space and an absolute time is vital

to the theory (for the potential at an assigned point and an assigned instant is calculated from the *simultaneous* positions of the gravitating bodies, and so gravitation is regarded as an influence which is propagated at an infinite speed), it cannot be so in any law of gravitation consistent with the Relativity principle. This point can be further developed by reference to an approximate solution of his equation for a general distribution of energy, stress, and momentum obtained by Einstein in the form of a group of integrals resembling those which appear in classical electromagnetic theory as " retarded potentials." * From what has been stated in the preceding pages, it appears that in general the potentials at any point-instant differ from the Galilean values:

$$\begin{matrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}$$

by quantities of a small order of magnitude. Let us write

$$g_{\mu\nu} = \delta_{\mu\nu} + h_{\mu\nu},$$

where $\delta_{\mu\nu}$ is the Galilean value of the $\mu\nu$-component (so that $\delta_{\mu\nu} = 0$ if $\mu \neq \nu$; $\delta_{\mu\nu} = -1$ if $\mu = \nu = 1$, 2, or 3; $\delta_{\mu\nu} = 1$ if $\mu = \nu = 4$), and $h_{\mu\nu}$ is a quantity of the first order of magnitude. In what follows we shall neglect squares and products of the $h_{\mu\nu}$-functions, regarding them as quantities of the second order. To this approximation it can be shown by a short series of straightforward steps that

$$\begin{aligned} G_{\mu\nu} = & - \partial\{\mu\nu, \alpha\}/\partial x_\alpha + \partial\{\mu\alpha, \alpha\}/\partial x_\nu - \{\mu\nu, \alpha\}\{\alpha\beta, \beta\} \\ & \qquad + \{\mu\alpha, \beta\}\{\nu\beta, \alpha\} \\ = & - \tfrac{1}{2}\partial[g^{\alpha\beta}(\partial g_{\nu\beta}/\partial x_\mu + \partial g_{\mu\beta}/\partial x_\nu - \partial g_{\mu\nu}/\partial x_\beta)]/\partial x_\alpha \\ & \qquad + \tfrac{1}{2}\partial(g^{\alpha\beta}\partial g_{\alpha\beta}/\partial x_\nu)/\partial x_\mu. \end{aligned}$$

On the right-hand side we can replace $g^{\mu\nu}$ by $\delta^{\mu\nu}$ to the same degree of approximation, where $\delta^{11} = \delta^{22} = \delta^{33} = -1$ and $\delta^{44} = 1$ and the others are zero. We shall write $h_\mu{}^\nu$ for $g^{\nu\alpha}h_{\mu\alpha}$, or practically $\delta^{\nu\alpha}h_{\mu\alpha}$. Thus

$$h_\mu{}^1 = -h_{\mu 1},\ h_\mu{}^2 = -h_{\mu 2},\ h_\mu{}^3 = -h_{\mu 3},\ h_\mu{}^4 = h_{\mu 4}$$

and
$$h = h_\alpha{}^\alpha = -h_{11} - h_{22} - h_{33} + h_{44}.$$

It follows that

* Einstein, " Sitz. Preuss. Akad." (1916), pp. 688-696; *ibid.* (1918), pp. 154-167.

$$G_{\mu\nu} = \tfrac{1}{2}(\delta^{\alpha\beta}\partial^2 h_{\mu\nu}/\partial x_\alpha \partial x_\beta + \partial^2 h/\partial x_\mu \partial x_\nu - \partial^2 h_\nu{}^\alpha/\partial x_\mu \partial x_\alpha - \partial^2 h_\mu{}^\alpha/\partial x_\nu \partial x_\alpha)$$
$$= \tfrac{1}{2}(\Box h_{\mu\nu} + \partial^2 h/\partial x_\mu \partial x_\nu - \partial^2 h_\nu{}^\alpha/\partial x_\mu \partial x_\alpha - \partial^2 h_\mu{}^\alpha/\partial x_\nu \partial x_\alpha) \quad (36)$$

where we write $\Box$ for the operator

$$-\partial^2/\partial x_1{}^2 - \partial^2/\partial x_2{}^2 - \partial^2/\partial x_3 + \partial^2/\partial x_4{}^2.$$

Now define $f_{\mu\nu}$ by the equation

$$\begin{aligned} f_{\mu\nu} &= h_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}h \\ &= h_{\mu\nu} - \tfrac{1}{2}\delta_{\mu\nu}h \end{aligned} \quad . \quad . \quad . \quad (37)$$

and assume that $f_{\mu\nu}$ satisfies the four conditions expressed by the equation

$$\partial f_\mu{}^\alpha/\partial x_\alpha = 0. \quad . \quad . \quad . \quad . \quad (38)$$

If this assumption be satisfied, then we see from (36) that

$$G_{\mu\nu} = \tfrac{1}{2}\Box h_{\mu\nu},$$

and therefore

$$G_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}G = \tfrac{1}{2}\Box f_{\mu\nu}. \quad . \quad . \quad . \quad (39)$$

As a consequence of (39), Einstein's gravitational equations (22A of Chapter XI.) become

$$\Box f_{\mu\nu} = -16\pi\kappa T_{\mu\nu}. \quad . \quad . \quad . \quad (40)$$

The validity of these equations depends, of course, on the possibility of satisfying the four relations (38), and this is tantamount to choosing a special frame of reference. Einstein's general equations (22) or (22A) of Chapter XI. are covariant for any arbitrary transformation. On introducing a new co-ordinate system the $g_{\mu\nu}$-potentials of the new system depend on four arbitrary functions which define the transformation of co-ordinates. These four functions can be chosen in such a manner that the $g_{\mu\nu}$ of the new system can satisfy any assigned four conditions. Thus for our present purpose these four functions must be chosen in such a manner that the $g_{\mu\nu}$ or $h_{\mu\nu}$ satisfy (38). That is, by choosing a special co-ordinate system, we can ensure the truth of equation (40) to the first degree of approximation. Those who are familiar with the solution of the wave equation in electromagnetic theory in terms of retarded potentials, will recognise at once that the solution of (40) is

$$f_{\mu\nu} = -4\kappa\iiint [T_{\mu\nu}]/r \, . \, d\tau. \quad . \quad . \quad . \quad (41)$$

In (41) $d\tau$ is a volume element of the three-dimensional

space (for which x_1, x_2, x_3 represent point co-ordinates) at a distance r from the point where $f_{\mu\nu}$ is being estimated, and $[T_{\mu\nu}]$ is the value of the component of the matter tensor not at the instant x_4, when $f_{\mu\nu}$ is being estimated, but at the instant $x_4 - r$; that is, at an instant earlier by the interval of time required for light to travel from the element $d\tau$ to the point where $f_{\mu\nu}$ is being determined. This solution determines $g_{\mu\nu}$ also, for by (37) it follows that

$$\begin{aligned} f &= g^{\alpha\beta}h_{\alpha\beta} - \tfrac{1}{2}g^{\alpha\beta}g_{\alpha\beta}h \\ &= \delta^{\alpha\beta}h_{\alpha\beta} - 2h \\ &= -h, \end{aligned}$$

and so

$$\begin{aligned} h_{\mu\nu} &= f_{\mu\nu} - \tfrac{1}{2}\delta_{\mu\nu}f \\ &= -4\kappa\iiint[T_{\mu\nu}]/r\,.\,d\tau + 2\delta_{\mu\nu}\kappa\iiint[T]/r\,.\,d\tau \quad . \quad (42) \end{aligned}$$

and

$$g_{\mu\nu} = \delta_{\mu\nu} + h_{\mu\nu}.$$

One obvious conclusion is that gravitational changes should be propagated with the speed of light, but it must be very definitely borne in mind that this conclusion is valid under the two very restrictive limitations introduced above, viz., that we are considering the matter from the point of view of a specially chosen frame of reference, and we are neglecting squares, etc., of $h_{\mu\nu}$.

An important fact which Einstein discovered is that certain types of plane gravitational waves transport no energy, and so can hardly be said to have a physical existence. To understand the steps by which this conclusion can be reached, we recall that the Einstein equation of conservation, viz.,

$$(T_\mu{}^\alpha)_\alpha = 0,$$

or

$$\partial(qT_\mu{}^\alpha)/\partial x_\alpha + \tfrac{1}{2}qT_{\alpha\beta}\partial g^{\alpha\beta}/\partial x_\mu = 0$$

can be written in the form

$$\partial(T_\mu{}^\alpha + t_\mu{}^\alpha)/\partial x_\alpha = 0,$$

where $t_{\mu\nu}$ is defined by (10) of Chapter XI., *provided the system of co-ordinates chosen satisfies the condition $g = -1$.* Now the condition (38), which we have just been imposing on the co-ordinates, is entirely different from the condition $g = -1$, so we cannot draw the same conclusion concerning the expression of the conservation law without further investigation. The conclusion can clearly be justified to the approximation neces-

sary if we can obtain a new expression for $t_{\mu\nu}$ which, subject to condition (38), will satisfy

$$\begin{aligned}\partial t_{\mu}{}^{\alpha}/\partial x_{\alpha} &= \tfrac{1}{2}T_{\alpha\beta}\partial g^{\alpha\beta}/\partial x_{\mu} \\ &= \tfrac{1}{2}T_{\alpha\beta}\partial h^{\alpha\beta}/\partial x_{\mu} \\ &= -\,8\pi\kappa\,\Box f_{\alpha\beta}(\partial f^{\alpha\beta}/\partial x_{\mu} - \tfrac{1}{2}\delta^{\alpha\beta}\partial f/\partial x_{\mu}) \\ &= 8\pi\kappa\delta^{\gamma\epsilon}\partial^{2}f_{\alpha\beta}/\partial x_{\gamma}\partial x_{\epsilon}\,.\,(\partial f^{\alpha\beta}/\partial x_{\mu} - \tfrac{1}{2}\delta^{\alpha\beta}\partial f/\partial x_{\mu}).\end{aligned}$$

To the degree of approximation involved it can be shown that the following expression for $t_{\mu\nu}$ will satisfy this equation:

$$\begin{aligned}32\pi\kappa t_{\mu\nu} &= \partial f_{\alpha}{}^{\beta}/\partial x_{\mu}\,.\,\partial f_{\beta}{}^{\alpha}/\partial x_{\nu} - \tfrac{1}{2}\partial f/\partial x_{\mu}\,.\,\partial f/\partial x_{\nu} \\ &\quad - \tfrac{1}{2}\delta_{\mu\nu}\delta_{\gamma\epsilon}(\partial f_{\alpha}{}^{\beta}/\partial x_{\gamma}\,.\,\partial f_{\beta}{}^{\alpha}/\partial x_{\epsilon} - \tfrac{1}{2}\partial f/\partial x_{\gamma}\,.\,\partial f/\partial x_{\epsilon}), \quad . \quad (43)\end{aligned}$$

where

$$f_{\mu}{}^{\nu} = \delta^{\nu\alpha}f_{\mu\alpha},$$

so that

$$f_{\mu}{}^{1} = -f_{\mu 1},\ f_{\mu}{}^{2} = -f_{\mu 2},\ f_{\mu}{}^{3} = -f_{\mu 3},\ f_{\mu}{}^{4} = f_{\mu 4}.$$

To the same degree of approximation

$$\begin{aligned}T_{\mu\nu} &= \delta_{\mu\alpha}\delta_{\nu\beta}T^{\mu\nu} \\ &= \delta_{\mu\alpha}\delta_{\nu\beta}\rho dx_{\mu}/ds\,.\,dx_{\nu}/ds,\end{aligned}$$

and thus T_{41}, T_{42}, T_{43} are numerically equal to the components of the density of the energy-stream within matter. So t_{41}, t_{42}, t_{43} represent the density of the energy-stream in the gravitational field.

In the case of a plane wave propagated along the axis of x_1 with unit velocity, we have

$$f_{\mu\nu} = a_{\mu\nu}\,\phi(x_1 + x_4),$$

where the $a_{\mu\nu}$ are ten constants and ϕ is a function of the argument $x_1 + x_4$. As we are using the specially chosen system of co-ordinates defined by (38), it follows easily that

$$\left.\begin{aligned} a_{11} + a_{14} &= 0 \\ a_{21} + a_{24} &= 0 \\ a_{31} + a_{34} &= 0 \\ a_{41} + a_{44} &= 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (44)$$

By means of (43) and (44) we can determine the energy stream components. They are

$$\begin{aligned} t_{41} &= (32\pi\kappa)^{-1}\{[(a_{22} + a_{33})^2/4 + a_{23}{}^2]\phi'\}^2 \quad . \quad (45) \\ t_{42} &= t_{43} = 0. \end{aligned}$$

In this ϕ' represents the first derived function of ϕ with

respect to its argument. Following a classification by Weyl, we can group plane waves into three types:

Type 1 are those for which the $a_{\mu\nu}$ involving 1 or 4 in the suffix are zero. These are called "transverse-transverse" waves.

Type 2 are those for which the $a_{\mu\nu}$ are zero except a_{12}, a_{13}, a_{24}, a_{34}. These are called "longitudinal-transverse" waves.

Type 3 are those for which the $a_{\mu\nu}$ are zero, except a_{11}, a_{14}, a_{44}. These are called "longitudinal-longitudinal" waves.

It is clear from (45) that plane waves of types (2) and (3) transport no energy, and also waves of type (1), for which $a_{22} + a_{33} = 0$ and $a_{23} = 0$.

Addendum.

On account of the severe restriction on the choice of co-ordinates imposed by (38), the question of propagation in any natural system of co-ordinates is left rather undecided by Einstein's analysis. As this book goes to press a paper by Eddington ("Proc. R.S.," A 102, Dec. 1922) has just appeared, which treats the matter from a wider point of view. He considers first of all plane waves

$$h_{\mu\nu} = a_{\mu\nu}\phi(x_1 + Vx_4),$$

where the $a_{\mu\nu}$ are constants, V is a velocity of propagation, and the co-ordinates are general. These are propagated in the negative direction of the axis of x_1. Using the same approximations as before, he works out the components of Riemann-Christoffel tensor, and from them those of the Einstein tensor $G_{\mu\nu}$. Equating the latter to zero, he finds that such values of the $g_{\mu\nu}$ potentials can exist in the field provided the following seven equations are satisfied:

$$(1 - V^2)a_{22} = 0 \qquad (46)$$

$$(1 - V^2)a_{33} = 0 \qquad (47)$$

$$(1 - V^2)a_{23} = 0 \qquad (48)$$

$$a_{22} + a_{33} = 0 \qquad (49)$$

$$a_{24} = Va_{12} \qquad (50)$$

$$a_{34} = Va_{13} \qquad (51)$$

$$a_{44} - 2Va_{14} + Va_{11}{}^2 = 0. \qquad (52)$$

We classify these waves into three groups as before. Each group represents a disturbance which can be propagated independently of the others. Equation (49) shows that for a TT wave a_{22}, a_{33}, a_{23} cannot all vanish; hence $V = 1$, or TT plane waves are propagated with the speed of light. If, however, LT or LL waves exist, equations (50), (51), and (52) show that the value of V in each case depends on the coefficients of the disturbance, and exhibits no sign of approximating to the velocity of light. These waves have therefore no fixed velocity and, as Einstein discovered, they transport no energy. They have, in fact, no objective existence, and cannot be detected by any conceivable experiment. By his special choice of co-ordinates, Einstein, as it were, compels these spurious waves to travel with the same velocity as the genuine TT waves. "He imposes conditions on the co-ordinates which bar out most of the spurious disturbances possible; but those which have the velocity of light take advantage of their close resemblance to the genuine waves and slip through the barrier" (Eddington, *loc. cit.*). In short, these spurious waves "are merely sinuosities in the co-ordinate system, and the only speed of propagation relevant to them is the speed of thought."

As a matter of fact, an investigation of the analysis shows that the Riemann-Christoffel tensor depends only on the differential coefficients of h_{22}, h_{33}, and h_{23} with respect to the argument, $x_1 + Vx_4$, and so the tensor depends entirely on the TT waves. For LT and LL waves the tensor vanishes and space-time is flat; in other words, the "supposed disturbance is an analytical fiction."

A remarkable conclusion which Eddington draws from the condition

$$h_{22} + h_{33} = 0,$$

satisfied by the TT waves, is the fact that these waves arise from very secondary disturbances in the distribution of matter, such as periodic stresses occurring in an infinite plate, or a circular motion of the plate in its own plane. A backward and forward motion of an *infinite* plate produces no waves, although similar behaviour of a *small* plate would. (They would be divergent and not plane.) Variations of T_{44}, i.e. alternate creation and destruction of mass, would likewise be ineffective, for "nature, having made no provision for the propagation of the corresponding disturbance, thereby automatically prevents the construction of such a source."

But plane waves are too artificial, and in order to illustrate

fully propagation of gravitational potential, divergent waves have to be studied. This Eddington proceeds to do, making use of Einstein's solution (42), obtained above, and employing, of course, the restricted choice of co-ordinates. The reader is referred to the paper for the details, but the following broad conclusions can be stated briefly.

Despite the resemblance between the equations of propagation of the $g_{\mu\nu}$ and the equations in the theory of sound, simple isotropic spherical waves of gravitational disturbance cannot occur. The reason for this is the fact that if ϕ is the potential in the acoustic problem,

$$\Box \phi = 0$$

unconditionally, while in the gravitational problem

$$\Box g_{\mu\nu} = 0$$

provided (38) is satisfied, i.e., provided the sources obey the conservation of energy and momentum. This bars out at once such sources as points where matter is alternately created and destroyed, from which simple spherical waves might be expected to arise. In cases where conservation is obeyed, as, for instance, in the case of a rod spinning round an axis perpendicular to itself (a problem first treated by Einstein), while simple spherical waves occur for some components of $g_{\mu\nu}$, they must be accompanied by waves arising from a *doublet* source for other components.

Spinning bodies produce divergent waves, and slowly lose energy by reason of them, a result first discovered by Einstein.

Spurious divergent waves can exist just like spurious plane waves, but they cannot be readily distinguished from the genuine, and Weyl's classification into three types which can exist independently of one another no longer holds.

Types of TT waves which are inconsistent with the equations

$$G_{\mu\nu} = 0$$

exist; that is, other physical manifestations not of a purely gravitational character accompany them. For instance, we can have TT waves for which $h_{22} + h_{33}$ is not zero. They can exist in space which, although not empty in a strict sense, is empty of matter. In short, energy of radiation is present, and propagation of light accompanies the propagation of gravitation in the case of these " electromagnetic gravitational waves," as Eddington calls them

PART III.

WORLD GEOMETRY.

CHAPTER XIV.

ALTHOUGH the Relativity theory has been the direct product of physical experiment, yet the appeal to geometrical ideas connected with a conceptual four-dimensional world which cannot be grasped intuitively has been manifest throughout the preceding pages. While removing space and time from the category of the absolute, experiment lent support to the introduction of a new absolute, Minkowski's element of separation between two events. The first step which Einstein took in framing his theory of Gravitation consisted in a generalisation of the mathematical expression for an element of separation. In this step and in his suggestion that gravitational orbits in any frame of reference are the projections of " natural " world-lines or " geodesic tracks " in space-time, he evaded a criticism which had been levelled at Newton's theory by his great contemporaries Huygens and Leibnitz, and which in one form or another has occupied the minds of scientific thinkers ever since. While every one had to admit the remarkable power of the Newtonian formula for gravitational force in summarising the astronomical evidence, it was felt that the idea of " action at a distance " constituted a serious breach of a principle which impressed itself with great effect on scientific opinion, especially during the nineteenth century—the principle of Continuity. How could two bodies separated by a finite extension in space affect each other's movements? When Maxwell had shown that the theories of action at a distance employed in the electrostatic and electrokinetic theory of the early nineteenth century had to be replaced by the continuous action through a medium implied in his electromagnetic theory and explicitly expressed in the *differential* form of his equations of the field, it was only natural that similar ideas concerning a possible transmission of gravitational action by the same medium from body to body should impress themselves on scientific opinion with renewed force. One serious difficulty, however, blocked the way. It was generally believed on the weighty authority of Laplace that such an action would have to be transmitted with an infinite

speed in order to agree with the observed motions of the planets, and such a conclusion was felt to be somewhat at variance with the finite speed of propagation of radiation; indeed, the difficulty bore some resemblance to the famous troubles which surrounded the question of the "longitudinal wave" in the elastic-solid theories of the ether, and it was only the investigations of Lorentz and Poincaré on the possibility of generalising Newton's formula to fit the Relativity principle which revealed the fallacy in Laplace's reasoning.

It is clear, however, at the outset that Einstein's theory involves no violation of the Continuity principle, for the simple reason that no gravitational force is invoked at all. In Newton's theory the "natural" paths in *absolute space* are straight; force is that which deflects bodies from these paths, and so gravitational force has to be postulated in addition to mechanical and electromagnetic forces. In Einstein's theory there are in a definite frame of reference "natural" paths given by a certain differential equation; "force" is that which deflects bodies from these paths, but it is purely mechanical or electromagnetic in character. The idea of continuity is satisfied because the "natural" paths are one aspect of the world-lines or geodesic tracks which connect, element by element, two events in the universe separated by a finite extension in space-time. But this procedure involves of necessity an appeal to geometric ideas, which, truth to tell, does not meet with great favour in some quarters, and criticisms are from time to time launched at what is called the "geometrisation of Physics." It is to this feature that we must now give closer attention, and it will be necessary to indicate the connection which exists between Einstein's theory and the famous researches in Differential Geometry of Gauss, Riemann, Christoffel, and Levi-Cività. The mind of the reader will thus be prepared for a consideration of the problem of boundary conditions which is engaging the attention of relativists at the moment and leading to rather startling conclusions concerning the scale of the universe, and he will also be enabled to grasp a further generalisation of the Relativity theory by Weyl, which seems destined to relegate electromagnetic force to the same position as that into which gravitational force has been thrust.

Nothing brings home the geometrical nature of Einstein's theory more forcibly perhaps than the constant recurrence of the phrase, "curvature of space-time." It is meaningless to those who are not only unable to visualise such a thing (forming no exception among men in that respect), but are also unaware

that it is a name for a definite analytical formula. It will shed some light on this concept and its origin if we give a brief summary of Gauss' researches on curvature of surfaces and Riemann's extension of them to n-dimensional manifolds.

Metric Properties and Curvatures of Surfaces.

In the first place, if we choose a point O on a surface as origin of co-ordinates, and the tangent plane at O as the plane $z = 0$, the equation of the surface can be written in the form,

$$z = ax^2 + 2hxy + by^2 + \ldots$$

the remaining terms being of the third order at least in x and y.

If a point P be chosen on the curve in which the plane ZOX cuts the surface, and a normal *to the surface* be drawn at P, this normal does not in general cut the normal OZ. Even as P approaches O, the shortest distance between the normal at P and the normal OZ bears a *finite* ratio to the distance OP. Similar statements are in general true concerning the normal to the surface at a point Q on the section by ZOY. There is an exception to this general rule. By rotation of the rectangular axes OX, OY in the plane $z = 0$, it is possible to find a position of these axes for which the coefficient of the term in xy is zero, and the equation of the surface can be written

$$z = ax^2 + by^2 + \ldots \qquad (1)$$

It is known that in this case the ratio of the shortest distance between the normal at P and OZ to the distance OP approaches zero as P approaches O. In brief, the normal at a point on the section by ZOX adjacent to O meets the normal at O in a definite point C_1, called a centre of principal curvature. Similarly an adjacent point Q on the section by ZOY gives a second centre C_2. OP and OQ are elements at O of the two lines of principal curvature which cross at O. At every point on the surface there cross two such lines, and at every point there exist two principal centres. Such lines in fact form two orthogonal families of curves on the surface, and any finite number of them give a mesh configuration on the surface. Also, it can be proved that if $R_1 = C_1O$, $R_2 = C_2O$ (the principal radii of curvature at O), then the coefficients a and b in (1) are $1/2R_1$ and $1/2R_2$ respectively. Now let us draw through P the other line of curvature at P, and through Q the other line of curvature at Q, these meeting in S (say); then OPSQ is an elementary rectangle on the surface whose area is

$$R_1R_2\delta\theta_1\delta\theta_2,$$

where $\delta\theta_1 = \angle OC_1P$, and $\delta\theta_2 = \angle OC_2Q$.

If from any point C four lines are drawn parallel to the normals at O, P, Q, S, these lines cut out on the surface of a sphere of radius R drawn with C as centre an elementary rectangle of area $R^2\delta\theta_1\delta\theta_2$. Suppose R is so chosen that the two rectangles are equal, so that $R = (R_1R_2)^{\frac{1}{2}}$, the surface is said to have at the point O the same curvature as the sphere. The "measure of curvature" of the sphere is $1/R^2$, and so $1/R_1R_2$ has been named by Gauss the measure of curvature of the surface at the point O.

Abandoning these special axes, suppose we choose any arbitrary Cartesian co-ordinates, and make x, y, z functions of two independent variables x_1 and x_2, so that

$$\begin{aligned} x &= \xi(x_1, x_2) \\ y &= \eta(x_1, x_2) \\ z &= \zeta(x_1, x_2). \end{aligned}$$

Any definite pair of values assigned to x_1 and x_2 determines one point on a surface, the surface being that whose equation is obtained by the elimination of x_1 and x_2 from the three equations written. (Thus, if

$$\begin{aligned} x &= a \cos x_1 \\ y &= b \sin x_1 \cos x_2 \\ z &= c \sin x_1 \sin x_2, \end{aligned}$$

the surface is the ellipsoid

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1.)$$

When a definite value is assigned to x_2, and x_1 varies arbitrarily the point is restricted to a definite curve on the surface, thus

$$x_2 = \text{constant}$$

is a family of curves, and

$$x_1 = \text{constant}$$

is another family of curves, which in general mark out on the surface a mesh configuration. (It is not necessarily a rectangular mesh, nor need these lines be lines of principal curvature.)

As an abbreviation for the first and second differential

coefficients of x, y, and z with respect to x_1 and x_2, we shall employ the following convenient notation:—

$$\partial\xi/\partial x_1 = a_1\,;\ \ \partial\xi/\partial x_2 = a_2$$
$$\partial\eta/\partial x_1 = b_1\,;\ \ \partial\eta/\partial x_2 = b_2$$
$$\partial\zeta/\partial x_1 = c_1\,;\ \ \partial\zeta/\partial x_2 = c_2$$
$$\partial a_1/\partial x_1 = a_{11}\,;\ \ \partial a_1/\partial x_2 = a_{12} = a_{21} = \partial a_2/\partial x_1\,;\ \ \partial a_2/\partial x_2 = a_{22}$$

and a similar notation in $b_{11}, \ldots, c_{22}$ for the remainder.

All these are, of course, functions of x_1 and x_2, and are not in general constant over the surface.

Thus if (x_1, x_2) and $(x_1 + \delta x_1, x_2 + \delta x_2)$ are neighbouring points on the surface, the square of their distance apart is equal to

$$\left.\begin{aligned} &\delta x^2 + \delta y^2 + \delta z^2 \\ &= \Sigma(a_1\delta x_1 + a_2\delta x_2)^2 \\ &= g_{11}\delta x_1^2 + g_{22}\delta x_2^2 + 2g_{12}\delta x_1\delta x_2 \end{aligned}\right\} \quad . \quad . \quad (2)$$

where

$$g_{11} = \Sigma a_1^2$$
$$g_{22} = \Sigma a_2^2$$
$$g_{12} = \Sigma a_1 a_2 = g_{21}.$$

Moreover, since it is easily proved that

$$(b_1c_2 - b_2c_1)\delta x + (c_1a_2 - c_2a_1)\delta y + (a_1b_2 - a_2b_1)\delta z = 0$$

it follows that $b_1c_2 - b_2c_1$, $c_1a_2 - c_2a_1$, $a_1b_2 - a_2b_1$, are proportional to the direction cosines of the normal to the surface at the point (x_1, x_2). Now

$$\begin{aligned} \Sigma(b_1c_2 - b_2c_1)^2 &= \Sigma a_1^2\Sigma a_2^2 - (\Sigma a_1a_2)^2 \\ &= g_{11}g_{22} - g_{12}^2 \\ &= g \end{aligned}$$

where g is the determinant

$$\begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix}$$

so that the direction cosines are

$$(b_1c_2 - b_2c_1)/\sqrt{g}, \text{ etc.}$$

Consider the elementary parallelogram on the surface whose corners are the points A, B, C, D with the co-ordinates

$$(x_1, x_2)\ \ (x_1 + \delta x_1, x_2)\ \ (x_1 + \delta x_1, x_2 + \delta x_2)\ \ (x_1, x_2 + \delta x_2)$$

respectively. Since

$$AB = \surd(g_{11}\delta x_1^2)$$
$$AD = \surd(g_{22}\delta x_2^2)$$
and
$$AC = \surd(g_{11}\delta x_1^2 + g_{22}\delta x_2^2 + 2g_{12}\delta x_1\delta x_2),$$

it follows that

$$\cos \angle BAC = g_{12}/\surd(g_{11}g_{22}),$$

and therefore

$$\sin \angle BAC = \surd(g/g_1g_2).$$

In consequence the area of this parallelogram is $g^{\frac{1}{2}}\delta x_1\delta x_2$, and therefore the area of the part of the surface bounded by the four lines

$$x_1 = h_1,\ x_2 = h_2,\ x_1 = k_1,\ x_2 = k_2$$

is

$$\iint g^{\frac{1}{2}}dx_1dx_2 \quad . \quad . \quad . \quad . \quad . \quad (3)$$

the domain of integration being $h_1 \leq x_1 \leq k_1$, $h_2 \leq x_2 \leq k_2$.

The results (2) and (3) show that the mensuration of lengths and area on the surface are directly dependent on the values of the three coefficients g_{11}, g_{22}, g_{12}.

Gauss demonstrated a still more striking conclusion, viz., that the curvature of the surface at a point is a definite function of g_{11}, g_{22}, g_{12} and their differential coefficients with respect to x_1 and x_2, and in so doing introduced the prototype for two dimensions of the Riemann-Christoffel tensor.

The proof of this result is somewhat tedious and involved as obtained by Gauss, and so to avoid a long digression, it is given in an appendix. Gauss succeeded in showing that the measure of curvature is

$$R_{1212}/g \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where we are employing the R_{1212} symbol defined as in Chapter IX. to be

$$\partial[11, 2]/\partial x_2 - \partial[12, 2]/\partial x_1 + (\{12, a\}[12, a] - \{11, a\}[22, a]),$$

the summation for the dummy suffix being of course only for two variables.

We can use the properties of these symbols as explained in Chapter IX., to find the mixed tensors. Hence we can find the contracted tensors R_{11}, R_{22}, R_{12} (the summation in the contraction being, of course, only for $a = 1$ and $a = 2$. Finally, we calculate the invariant R, i.e., $g^{11}R_{11} + g^{22}R_{22} + 2g^{12}R_{12}$, which turns out to be

$$-2R_{1212}/g,$$

so that the curvature of the surface at the point is

$$-R/2. \quad . \quad . \quad . \quad . \quad (5)$$

There is another method, due to Levi-Cività, of dealing with the curvature of a surface, which dispenses with the use of normals to the surface, but is connected with what we may call an attempt to carry a vector with a given direction from one point on the surface to another. On a Euclidean plane the axiom of parallelism defines for us the same direction at two different points. The matter is not so easy on a curved surface. For simplicity, assume we are dealing with a spherical surface. Choose two points P and Q on it, and define a direction at P by an element of a great circle passing through P. Can we attribute any meaning to " the same direction " at Q? What great circle through Q (if any) possesses an element at Q which is " parallel " to the element at P. No circle certainly has this property in general if " parallel " is to be interpreted in accordance with *three-dimensional* Euclidean geometry. (Only if the great circle through P happened to be at right angles to the great circle passing through P and Q would that be possible.) But if we restrict our directions to be *tangential* to the sphere, we can select from all such directions at Q one which is uniquely related to the given direction at P, and which could be with convenience named as the " same " direction as that at P. Let us choose the great circle at Q, which makes with the great circle PQ an angle on the same side equal to that made by the great circle through P, and define the direction of its element at Q to be the " same " as the direction at P. This will certainly lead to no ambiguity nor to any contradiction, provided all directions considered are restricted to being tangential to the sphere. Now consider a spherical triangle PQR, and transfer in this manner a given direction at P to the " same " direction at Q (by what we may call a " parallel displacement "), then transfer the direction at Q to the " same " direction at R in the same manner along the great circle R. Now transfer the original direction at P to R directly along the great circle PR. The two directions at R do not agree, as anyone will readily recognise who is aware of the elementary fact of spherical trigonometry that the three angles of a spherical triangle together make up an angle greater than two right angles. Of course, there is no ambiguity as to which of the two directions at R is really the " same " as that at P. By our definition it is clearly the latter, and what we prove is that successive displacements along any path do not in general

arrive at the "same" direction at another point, but only the direct displacement along the great circle path. This fact can be stated in a still more instructive fashion. After having carried the direction from P to R, via Q, transfer it back to P directly along the great circle RP. The new direction at P does not agree with the original, but makes with it an angle equal to the excess of the sum of the angles of the spherical triangle above two right angles, and the sense of the rotation from the original direction to the new is right- or left-handed, according as we go round the triangle in a right- or left-handed sense. This fact is referred to as the non-integrability of direction. Now it is a well-known result that the value of this excess is equal to the quotient of the area of the triangle by the square of the radius of the sphere, i.e., the product of the area of the triangle and Gauss' measure of its curvature. The result can be easily generalised to any figure on the spherical surface. Indeed, it may be extended to any curved surface, provided we substitute the word "geodesic" for "great circle" in the above, and employ the "average measure of curvature" over the area bounded by the curve on the surface. In fact, this method can be employed to obtain the analytical result for Gauss' measure obtained above. We shall not stop to indicate the steps involved, as we shall presently be applying the method of "parallel displacement" to the more general case of an n-dimensional manifold.

Properties of a Manifold.

The curvature of the surface of a material body is a property which we perceive because the space of our experience is three-dimensional. The result just obtained is important in that it demonstrates that the analytical measure of that curvature, however, depends on the functions which help to express the distance between two points on the surface when a system of co-ordinates or mesh configuration has been defined, i.e., on the inner metrical relations of points on the surface quite apart from the relations between these points and points not on the surface. We are accustomed to speak of a surface as "plane," if we do not *perceive* any curvature; but it does not therefore follow that the Gaussian measure of curvature of a "plane" surface vanishes. If we assume that the postulates and axioms of Euclid (in particular his axiom of parallels) are true for the surface, then it can be shown very easily that the measure of curvature is zero, and our actual experience of material bodies

is such that these axioms and postulates, if not absolutely true, represent a very close approximation to the truth. But if we replace this set of axioms by another set which is assumed to be valid for a plane surface, the measure of curvature in Gauss' sense may not be zero—it is not zero, for example, in Lobatchewski's Geometry, which is founded on Euclid's postulates and axioms with the exception that Euclid's axiom of parallels is replaced by the hypothesis that through a given point in a plane there can be drawn not one but an infinity of lines which do not meet a given line in the plane, no matter how far produced in either direction. In this Geometry the metrical relations between the points on a plane are of such a nature that the function R in (5) is positive, and so the measure of curvature is negative. In fact, as those acquainted with non-Euclidean Geometries will know, the Euclidean geometry of figures constructed with geodesic lines on the anticlastic surface, known as the pseudo-sphere, is identical with the Lobatchewskian geometry of rectilinear figures on a plane.

It is only natural for the inquiring mind which has been put in possession of these facts to speculate on the possibility of a curvature in our three-dimensional space. Intuitive apprehension of such a curvature is apparently beyond us in the present state of human development, but an intellectual grasp of its possible existence is within the power of anyone who is capable of realising that the inner metrical relations of the space of our experience may not be just precisely those which are deduced in Euclid's Geometry. Nor need we stop at a curvature in three dimensions; speculation soon extends to the possibility of importing similar ideas into the treatment of any manifold, an n-dimensional manifold being a class of entities each one of which is uniquely defined by a group of n numbers. A curve, for example, is a one-dimensional manifold in which the entities are points; a surface is a two-dimensional manifold of which the component entities may be taken as points or curves; a volume is a three-dimensional manifold. The simple tones are a two-dimensional manifold in which the numbers refer to pitch and intensity. The class of entities which we call colours form a three-dimensional manifold if we accept the theory that any colour can be imitated by blending three primary colours in suitable proportions. If we consider human beings in five limited characteristics—height, span of arms, girth, weight, age—we would be dealing with a five-dimensional manifold. The latter would be a "discrete" manifold in contradistinction to the others mentioned where

there is continuity between the members of the class. Now we can always represent any manifold of dimensions not higher than three by spatial diagrams, each member of the class corresponding to a point in a curve, a surface or a volume, as the case may be. Such representation is not visually possible to us beyond three dimensions; nevertheless, the mathematical analysis of certain manifolds which are of importance in Physical Science is of such a nature that it develops formulæ and equations in four or more variables precisely analogous to formulæ and equations in three variables occurring in analytical geometry. The best example of this is the representation of the motion of a dynamical system with n degrees of freedom by a curve in $2n$-dimensional space, in which the geometrical co-ordinates are the Lagrange generalised co-ordinates of position and momentum of the system.

A little reflection, however, shows that in the representation of a one-, two-, or three-dimensional manifold by a space diagram, it may happen that some important property of space itself may be of no importance whatever in the representation, i.e., there may be no physical property of the manifold corresponding to this property of space. Thus in the representation of tones or colours in the manner referred to above no physical significance can be attached to the distance between two points on the diagrams, nor to the direction of the line joining them. The manifolds are, in fact, neither "metric" nor "affine" (to employ the phraseology of Levi-Cività and Weyl). Now we have seen that in the Relativity treatment of the space-time continuum, a quadratic function of the differentials of the co-ordinates is of great and fundamental importance, being the square, in fact, of the invariant "separation," and is the analogue of the quadratic function in the geometry of surfaces which represents the square of the invariant distance. So space-time is a metric four-dimensional manifold. Direction can also be postulated as a property of it; hence it is affine. Let us therefore consider any affine and metric n-dimensional manifold, i.e., a manifold for which there exist physical properties whose magnitudes are directly related to a step or displacement from one member of the manifold to a neighbouring member, and to some quadratic function of the differences of the co-ordinates of the two members, which maintains an invariant value for all modes of co-ordinating the members. If now we suppose such a manifold represented in an n-dimensional space, we can assume that such a step is represented by an elementary displacement in the space, and

that the physical property corresponding to the quadratic function is represented by the distance between two representative points. If this quadratic function has constant coefficients the space is " homaloidal " or flat ; if the coefficients are functions of the co-ordinates, ***and it is impossible to transform to a co-ordinate system in which the co-efficients are constants***, the space is " non-homaloidal," it is warped or curved in a space of higher dimensions, and we can arrive at an analytical expression for its curvature in the manner suggested above. Consider an elementary displacement $(A^1, A^2, \ldots, A^n)$ at a point $(x_1, x_2, \ldots, x_n)$. There is some displacement $(B^1, B^2, \ldots, B^n)$ at the point $(y_1, y_2, \ldots, y_n)$ which is equal to the former, i.e., has the same direction and magnitude ; but it is not in general one given by $B^\lambda = A^\lambda$. For example, if it happened to be so for one system of co-ordinates, it would not be so for others. Thus, transforming to accented co-ordinates, we would have

$$\left.\begin{aligned} A'^\lambda &= a_{\lambda\alpha}(x)A^\alpha \\ B'^\lambda &= a_{\lambda\alpha}(y)B^\alpha \end{aligned}\right\} \text{ (the summation is from } \alpha = 1 \text{ to } \alpha = n)$$

and since $a_{\lambda\mu}$ has not in general the same value at $(x_1 \ldots x_n)$ as at $(y_1 \ldots y_n)$, it follows that $A'^\lambda \neq B'^\lambda$, even though $A^\lambda = B^\lambda$. It is clear, therefore, that a displacement at the point y_λ, " parallel to " a displacement at x_λ, is not obtained as readily as one might suppose when the space is curved. Even if the space were flat equal elementary changes in the co-ordinates do not necessarily give equal displacements at different points as consideration of polar co-ordinates will promptly show.

Yet there is no meaning in saying that a manifold is affine unless there is a one-one correspondence between the directions at a point in the representative space and the directions at any other point. We have seen how such a correspondence could be set up geometrically in a two-dimensional curved space, and we can make the attempt by analysis in spaces of higher dimensions. Thus if P and Q are two neighbouring points x_λ and $x_\lambda + \delta x_\lambda$, and if the elementary displacement A^λ at P is equal to the elementary displacement B^λ at Q, then

$$B^\lambda = A^\lambda - f_{\alpha\beta}{}^\lambda A^\alpha \delta x_\beta, \quad . \quad . \quad . \quad (6)$$

where the summation for α and β extends from 1 to n for each, so that there are really n^2 terms represented by the second term on the right-hand side, and where $f_{\lambda\mu}{}^\nu$ are n^3 coefficients, which are definite functions of the co-ordinates, whose form depends

on the nature of the manifold or representative space, and will presently be discovered. The reader must be warned that $f_{\lambda\mu}{}^{\nu}$ is not a tensor (mixed of the third order), and so it is not written with a capital letter; but it is convenient to use a notation in which two indexes in the covariant place mechanically cancel, as it were, two in the contravariant place leaving one in the contravariant place in agreement with the other terms.* It might appear at first sight wrong to assert that $f_{\lambda\mu}{}^{\nu}$ is not a mixed tensor, for in (6) is not $f_{\alpha\beta}{}^{\lambda}A^{\alpha}\delta x_{\beta}$ a vector, being the difference of two vectors? This line of reasoning overlooks the fact, however, that A^{λ} and B^{λ} are estimated *at different points*, and so in transforming to another system of co-ordinates different values of $a_{\lambda\mu}$ would be involved in each case, and $f_{\alpha\beta}{}^{\lambda}A^{\alpha}\delta x_{\beta}$ would not transform as a vector.

We can justify (6), since the second term of the right-hand side is the most general linear expression for the changes in the components corresponding to displacement of the vector "without absolute change." The justification can be made more convincing by employing a postulate of Weyl's that there exists at each point a co-ordinate system which is "geodesic" for the immediate neighbourhood, i.e., one in which the components of a vector at the point do not vary when the vector is displaced without change to a neighbouring point. Let ξ_{λ} be this system, and let X^{λ} represent the vector at ξ_{λ} in this system. Let the transformation equations from the x co-ordinates to the ξ co-ordinates be given by

$$\delta\xi_{\lambda} = \phi_{\lambda\alpha}\delta x_{\alpha}$$

and

$$\delta x_{\lambda} = \psi_{\alpha\lambda}\delta\xi_{\alpha},$$

where $\phi_{\lambda\mu}$ and $\psi_{\lambda\mu}$ are functions of the co-ordinates such that

$$\begin{aligned}\phi_{\lambda\alpha}\psi_{\mu\alpha} &= 1 \text{ if } \lambda = \mu \\ &= 0 \text{ if } \lambda \neq \mu.\end{aligned}$$

(See Theorem I. of Chapter VIII.)

Hence

$$A^{\lambda} = \psi_{\gamma\lambda}X^{\gamma} \qquad \text{(value at the point P)}$$

and

$$B^{\lambda} = (\psi_{\gamma\lambda} + \delta\psi_{\gamma\lambda})X^{\gamma} \qquad (\;,,\;\;,,\;\;,,\;\;,,\;\; Q)$$

so

$$\begin{aligned}B^{\lambda} - A^{\lambda} &= \partial\psi_{\gamma\lambda}/\partial x_{\beta} \,.\, X^{\gamma}\delta x_{\beta} \\ &= \phi_{\gamma\alpha}\partial\psi_{\gamma\lambda}/\partial x_{\beta} \,.\, A^{\alpha}\delta x_{\beta}\end{aligned}$$

* Of course, the β in δx_{β} is actually to be considered in the contravariant position; in fact, some writers use for the sake of uniformity the notation $(x)^{\lambda}$, $(\delta x)^{\lambda}$ instead of x_{λ}, δx_{λ}.

which agrees with (6) if we put

$$f_{\mu\nu}{}^{\lambda} = -\phi_{\alpha\mu}\partial\psi_{\alpha\lambda}/\partial x_{\nu}.$$

If we take A^{λ} to represent an elementary displacement PR $= dx_{\lambda}$, which on parallel displacement to Q becomes QS, then the co-ordinates of S differ from those of P by

$$dx_{\lambda} + \delta x_{\lambda} - f_{\alpha\beta}{}^{\lambda}dx_{\alpha}\delta x_{\beta}.$$

If we now consider the displacement PQ displaced parallel to itself along PR into the position RS′, then the co-ordinates of S′ differ from those of P by

$$\delta x_{\lambda} + dx_{\lambda} - f_{\alpha\beta}{}^{\lambda}\delta x_{\alpha}dx_{\beta},$$

which can be written by a modification of dummy indices,

$$\delta x_{\lambda} + dx_{\lambda} - f_{\beta\alpha}{}^{\lambda}\delta x_{\beta}dx_{\alpha}.$$

It is a necessary condition for an affine geometry that S and S′ should agree to the first order, i.e., that the space should be flat to this order. This requires the condition

$$f_{\mu\nu}{}^{\lambda} = f_{\nu\mu}{}^{\lambda} \quad . \quad . \quad . \quad . \quad (7)$$

which can as a matter of fact be deduced from the expression given above for $f_{\mu\nu}{}^{\lambda}$.

The change in the components of a *covariant* vector which has suffered a parallel displacement can now be obtained readily. We can define a parallel displacement in this case by the condition that if A_{λ} is a covariant vector, and A^{λ} any arbitrary contravariant vector, then the invariant product $A_{\alpha}A^{\alpha}$ is to have the same value after a parallel displacement of both vectors. Hence if B_{λ} is the displaced covariant vector at the point $x_{\lambda} + \delta x_{\lambda}$, we have

$$B_{\gamma}(A^{\gamma} - f_{\alpha\beta}{}^{\gamma}A^{\alpha}\delta x_{\beta}) = A_{\gamma}A^{\gamma}$$

or

$$(B_{\gamma} - A_{\gamma} - f_{\gamma\beta}{}^{\alpha}B_{\alpha}\delta x_{\beta})A^{\gamma} = 0,$$

interchanging dummy suffixes in the last term.

Hence, since A^{λ} is arbitrary,

$$B_{\lambda} = A_{\lambda} + f_{\lambda\beta}{}^{\alpha}B_{\alpha}\delta x_{\beta}$$

or

$$B_{\lambda} = A_{\lambda} + f_{\lambda\beta}{}^{\alpha}A_{\alpha}\delta x_{\beta} \quad . \quad . \quad . \quad (8)$$

since quantities of the second order are being neglected.

It is of great importance to be quite clear about these "spurious" changes in A^{λ} and A_{λ}, which accompany "no

absolute change" in the vectors. To repeat, if the scheme of transformation from unaccented to accented co-ordinates is

$$A'^{\lambda} = a_{\lambda\alpha}A^{\alpha}$$
$$A_{\lambda}' = b_{\lambda\alpha}A_{\alpha}$$

then

$$A'^{\lambda} - f'_{\alpha\beta}{}^{\lambda}A'^{\alpha}\delta x_{\beta}' = (a_{\lambda\alpha} + \delta a_{\lambda\alpha})(A^{\alpha} - f_{\beta\gamma}{}^{\alpha}A^{\beta}\delta x_{\gamma})$$

and $$A_{\lambda}' + f'_{\lambda\beta}{}^{\alpha}A_{\alpha}'\delta x_{\beta}' = (b_{\lambda\alpha} + \delta b_{\lambda\alpha})(A_{\alpha} + f_{\alpha\gamma}{}^{\beta}A_{\beta}\delta x_{\gamma}),$$

where $a_{\lambda\mu} + \delta a_{\lambda\mu}$, $b_{\lambda\mu} + \delta b_{\lambda\mu}$ are the values of the transformation coefficients at $x_{\lambda} + \delta x_{\lambda}$.

Now let A^{λ} be any contravariant vector function which has the value $A^{\lambda} + \delta A^{\lambda}$ at the neighbouring point $x_{\lambda} + \delta x_{\lambda}$. Then since the difference between two vectors *at the same point* is a vector, it follows that

$$\delta A^{\lambda} + f_{\alpha\beta}{}^{\lambda}A^{\alpha}\delta x_{\beta}$$

is a contravariant vector. But δx_{λ} is a contravariant vector. Therefore

$$\partial A^{\lambda}/\partial x_{\mu} + f_{\alpha\mu}{}^{\lambda}A^{\alpha} \quad . \quad . \quad . \quad (9)$$

are the components of a mixed tensor of the second order, covariant in μ, contravariant in λ.

Similarly it can be shown that

$$\partial A_{\lambda}/\partial x_{\mu} - f_{\lambda\mu}{}^{\alpha}A^{\alpha} \quad . \quad . \quad . \quad (10)$$

is a covariant tensor of the second order.

Comparison of these expressions with the covariant derivatives obtained in Chapter IX. suggests that $f_{\lambda\mu}{}^{\nu} = \{\lambda\mu, \nu\}$, and we shall verify this presently, although we shall see later that the verification implies a certain restriction on our geometry.

An important particular case arises when the vector to be displaced is dx_{λ}/ds, ds being the invariant distance between two neighbouring points, so that the "spurious" change in the components is

$$- f_{\alpha\beta}{}^{\lambda}\delta x_{\beta}dx_{\alpha}/ds.$$

If this vector be displaced not merely parallel to itself, but also "along" itself, we obtain successive elements of a geodesic line, whose equation is therefore

$$\left.\begin{aligned} &\delta(dx_{\lambda}/ds) = -f_{\alpha\beta}{}^{\lambda}\delta x_{\beta}dx_{\alpha}/ds \\ \text{or} \quad &d^2x/ds^2 + f_{\alpha\beta}{}^{\lambda}dx_{\alpha}/ds \,.\, dx_{\beta}/ds = 0 \end{aligned}\right\} \quad . \quad . \quad (11)$$

Having defined the meaning of a parallel displacement more

closely by means of these formulæ, we shall investigate the bearing of them on the metric properties of the manifold. The distance between two contiguous points of the representative n-dimensional space is the square root of a quadratic function

$$g_{\alpha\beta}\delta x_{\alpha}\delta x_{\beta},$$

and the magnitude of a contravariant vector A^{λ} is the square root of

$$g_{\alpha\beta}A^{\alpha}A^{\beta}.$$

If this vector experiences a parallel displacement, it does not undergo any change in magnitude, hence

$$(\partial g_{\alpha\beta}/\partial x_{\epsilon} \,.\, A^{\alpha}A^{\beta} - g_{\alpha\theta}A^{\alpha}f_{\gamma\epsilon}{}^{\beta}A^{\gamma} - g_{\alpha\beta}A^{\beta}f_{\gamma\epsilon}{}^{\alpha}A^{\gamma})\delta x_{\epsilon} = 0,$$

or on suitably modifying the dummy suffixes

$$(\partial g_{\alpha\beta}/\partial x_{\epsilon} - f_{\beta\epsilon\alpha} - f_{\alpha\epsilon\beta})A^{\alpha}A^{\beta}\partial x_{\epsilon} = 0,$$

where $f_{\lambda\mu\nu} = g_{\nu\alpha}f_{\lambda\mu}{}^{\alpha}$, and so also $f_{\lambda\mu}{}^{\nu} = g^{\nu\alpha}f_{\lambda\mu\alpha}$.

Since the variations δx_{λ} are arbitrary,

$$\begin{aligned} f_{\lambda\nu\mu} + f_{\mu\nu\lambda} &= \partial g_{\lambda\mu}/\partial x_{\nu} \\ f_{\mu\lambda\nu} + f_{\nu\lambda\mu} &= \partial g_{\mu\nu}\ \partial x_{\lambda} \\ f_{\nu\mu\lambda} + f_{\lambda\mu\nu} &= \partial g_{\nu\lambda}/\partial x_{\mu}. \end{aligned}$$

Hence, as $f_{\lambda\mu\nu} = f_{\mu\lambda\nu}$

$$\begin{aligned} f_{\lambda\mu\nu} &= \tfrac{1}{2}(\partial g_{\lambda\nu}/\partial x_{\mu} + \partial g_{\mu\nu}/\partial x_{\lambda} - \partial g_{\lambda\mu}/\partial x_{\nu}), \\ &= [\lambda\mu,\ \nu] \end{aligned}$$

and in consequence

$$f_{\lambda\mu}{}^{\nu} = g^{\nu\alpha}[\lambda\mu,\ \alpha] = \{\lambda\mu,\ \alpha\}. \qquad (12)$$

In addition to laying down an expression for the distance between two points, we also shall require to deduce an expression for the area of an element of a two-dimensional surface in order to make progress towards an analytical expression for "curvature."

Let P be a point whose co-ordinates are $(x_1, x_2, \ldots, x_n)$; Q a point whose co-ordinates are $(x_1 + dx_1,\ x_2 + dx_2, \ldots, x_n + dx_n)$; R a point $(x_1 + \delta x_1, x_2 + \delta x_2, \ldots, x_n + \delta x_n)$; and S a point $(x_1 + dx_1 + \delta x_1, x_2 + dx_2 + \delta x_2, \ldots, x_n + dx_n + \delta x_n)$. We can refer to the point $(x_1,\ x_2, \ldots, x_{\lambda} + dx_{\lambda}, \ldots, x_n)$ as the projection of Q on the "local" λ-axis, and the point $(x_1,\ x_2, \ldots, x_{\lambda} + dx_{\lambda}, \ldots, x_{\mu} + dx_{\mu}, \ldots, x_n)$ as the projection of Q on the "local" $\lambda\mu$-co-ordinate surface.

Now from the result (3), proved earlier, it is clear that the area of the elementary parallelogram whose corners are P and the projections of Q, R and S on the local $\lambda\mu$-co-ordinate surface is

$$|\lambda\mu,\ \lambda\mu|^{\frac{1}{2}}(dx_\lambda \delta x_\mu - dx_\mu \delta x_\lambda) \quad . \quad . \quad . \quad (13)$$

where we write $|\kappa\lambda,\ \mu\nu|$ as a convenient symbol for the determinant

$$\begin{vmatrix} g_{\kappa\mu} & g_{\kappa\nu} \\ g_{\lambda\mu} & g_{\lambda\nu} \end{vmatrix}$$

This result can be generalised into the following theorem. The area of the elementary parallelogram PQRS is the square root of the sum of the n^4 terms

$$\tfrac{1}{4}|\,\alpha\beta,\ \gamma\delta\,|dS^{\alpha\beta}dS^{\gamma\delta} \quad . \quad . \quad . \quad (14)$$

where $dS^{\lambda\mu}$ is the anti-symmetrical tensor $dx_\lambda \delta x_\mu - dx_\mu \delta x_\lambda$, which corresponds to (but is not in general equal to) an element of area in the local $\lambda\mu$-surface. The expression (14) is of course a quadratic expression in the variables $dS^{\lambda\mu}$, whose coefficients are the determinants $|\kappa\lambda,\ \mu\nu|$.

To justify this result, we observe that as $dS^{\kappa\lambda}$ is a contravariant tensor of the second order, and $dS^{\kappa\lambda}dS^{\mu\nu}$ one of the fourth order, therefore

$$g_{\alpha\gamma}g_{\beta\delta}dS^{\alpha\beta}dS^{\gamma\delta}$$

is an invariant.

So also is

$$g_{\alpha\delta}g_{\beta\gamma}dS^{\alpha\beta}dS^{\delta\gamma}$$

an invariant.

Since $dS^{\gamma\delta} = -\,dS^{\delta\gamma}$, it follows that the expression (14) is an invariant. Its value is therefore unchanged if we transrorm to another set of co-ordinates, and in particular to one in which PQRS is the local 12-plane. In these (accented) co-ordinates all but four of the terms in the transformed expression (14) vanish. These terms correspond to

$$\begin{aligned} \alpha = \gamma = 1, &\quad \beta = \delta = 2 \\ \alpha = \delta = 1, &\quad \beta = \gamma = 2 \\ \alpha = \gamma = 2, &\quad \beta = \delta = 1 \\ \alpha = \delta = 2, &\quad \beta = \gamma = 1, \end{aligned}$$

and so the expression of (14) is equal to

$$|\,12,\ 12\,|' dS'^{12}dS'^{12},$$

which is the square of the area of PQRS.

CURVATURE OF A MANIFOLD.

We are now in a position to deal with the question of curvature in a manifold by the method suggested in connection with two-dimensional surfaces, viz., by the change produced in the direction of a vector by parallel displacement round a closed curve in the representative space.

Let a vector A^κ be carried by parallel displacement round a *small* complete circuit which embraces an area whose projections on the local co-ordinate planes correspond to the anti-symmetrical tensor $S^{\lambda\mu}$. Then if ΔA^κ represents the change in the κ-component of the vector after circulating round the circuit,

$$\Delta A^\kappa = \int \partial A^\kappa / \partial x_\beta \, . \, dx_\beta$$
$$= -\int f_{\beta\alpha}{}^\kappa A^\alpha dx_\beta \text{ by (6).}$$

Taking the terms under the integral in pairs and applying Stoke's Theorem we obtain

$$\Delta A^\kappa = \tfrac{1}{2}\iint[\partial(f_{\gamma\alpha}{}^\kappa A^\alpha)/\partial x_\beta - \partial(f_{\beta\alpha}{}^\kappa A^\alpha)/\partial x_\gamma]dS^{\beta\gamma}.$$

The integrand is by a further application of (6) equal to

$$A^\alpha(\partial f_{\gamma\alpha}{}^\kappa/\partial x_\beta - \partial f_{\beta\alpha}{}^\kappa/dx_\gamma) - (f_{\gamma\alpha}{}^\kappa f_{\beta\epsilon}{}^\alpha - f_{\beta\alpha}{}^\kappa f_{\gamma\epsilon}{}^\alpha)A^\epsilon,$$

which by a suitable modification of dummy suffixes is equal to

$$A^\alpha(\partial f_{\gamma\alpha}{}^\kappa/\partial x_\beta - \partial f_{\beta\alpha}{}^\kappa/\partial x_\gamma + f_{\gamma\alpha}{}^\epsilon f_{\beta\epsilon}{}^\kappa - f_{\beta\alpha}{}^\epsilon f_{\gamma\epsilon}{}^\kappa).$$

So that

$$\Delta A^\kappa = \tfrac{1}{2}\iint R_{\alpha\beta\gamma}{}^\kappa A^\alpha dS^{\beta\gamma} \quad . \quad . \quad (15)$$

where

$$R_{\lambda\mu\nu}{}^\kappa = \partial f_{\lambda\nu}{}^\kappa/\partial x_\mu - \partial f_{\lambda\mu}{}^\kappa/\partial x_\nu + f_{\lambda\nu}{}^\alpha f_{\alpha\mu}{}^\kappa - f_{\lambda\mu}{}^\alpha f_{\alpha\nu}{}^\kappa. \quad (16)$$

Since $f_{\lambda\mu}{}^\nu = \{\lambda\mu, \nu\}$, we see that $R_{\lambda\mu\nu}{}^\kappa$ is the Riemann-Christoffel mixed tensor which we met in Chapter IX., and equation (15) expresses its significance in the geometry ot multi-dimensional space. We can bring it into closer relationship with Gauss' result for a two-dimensional continuum as follows.

In the first place, we make use of the result that

$$g_{\alpha\beta}A^\alpha B^\beta$$

is the (invariant) geometric product (**A** . **B**) of the vectors **A** and **B**. This is a generalisation of the fact that $g_{\alpha\beta}A^\alpha A^\beta$ is the square of the invariant magnitude of the vector **A**. Then by (15)

$$(\Delta \mathbf{A} \,.\, \mathbf{B}) = \tfrac{1}{2}R_{\alpha\beta\gamma}{}^{\eta}g_{\eta\epsilon}A^{\alpha}B^{\epsilon}dS^{\beta\gamma},$$

where we consider the circuit as infinitesimal and drop the integral signs. Hence

$$(\Delta \mathbf{A} \,.\, \mathbf{B}) = \tfrac{1}{2}R_{\alpha\beta\gamma\epsilon}A^{\alpha}B^{\epsilon}dS^{\beta\gamma}.$$

Now let **A** and **B** be the vectors dx_λ and δx_λ represented by the sides PQ and PR of the elementary circuital parallelogram considered above. Now suppose that vector PQ, when transferred round the parallelogram, comes into the position PQ′, then Δ**A** is vector QQ′, and the magnitude of PQ is equal to the magnitude of PQ′. Thus in the limit QQ′ is perpendicular to PQ, and so the geometric product of Δ**A** and **B** is equal to the area of the parallelogram PQRS multiplied by the ratio of QQ′ to PQ, i.e., by the angle QPQ′, which measures the change in the direction of the vector **A** after its transference round the circuit.

Hence circuit area × vector deviation

$$= \tfrac{1}{2}R_{\alpha\beta\gamma\epsilon}dx_{\alpha}\delta x_{\epsilon}dS^{\beta\gamma}.$$

It is also equal to

$$\begin{aligned} &\tfrac{1}{2}R_{\epsilon\beta\gamma\alpha}dx_{\epsilon}\delta x_{\alpha}dS^{\beta\gamma} \\ = -&\tfrac{1}{2}R_{\alpha\beta\gamma\epsilon}\delta x_{\alpha}dx_{\epsilon}dS^{\beta\gamma}. \end{aligned}$$

Therefore it is equal to

$$\tfrac{1}{4}R_{\alpha\beta\gamma\epsilon}dS^{\alpha\epsilon}dS^{\beta\gamma}$$

or

$$\tfrac{1}{4}R_{\beta\gamma\epsilon\alpha}dS^{\alpha\beta}dS^{\gamma\epsilon}$$

(modifying the dummy indices).

Hence

$$\begin{aligned} \frac{\text{vector deviation}}{\text{circuit area}} &= \frac{\tfrac{1}{4}R_{\beta\gamma\epsilon\alpha}dS^{\alpha\beta}dS^{\gamma\epsilon}}{(\text{circuit area})^2} \\ &= \frac{R_{\beta\gamma\epsilon\alpha}dS^{\alpha\beta}dS^{\gamma\epsilon}}{|\,\alpha\beta,\ \gamma\epsilon\,|\,dS^{\alpha\beta}dS^{\gamma\epsilon}} \end{aligned}$$

or

$$= \frac{(\alpha\beta\gamma\epsilon)dS^{\alpha\beta}dS^{\gamma\epsilon}}{|\,\alpha\beta,\ \gamma\epsilon\,|\,dS^{\alpha\beta}dS^{\gamma\epsilon}} \quad . \quad . \quad (16)$$

in Riemann's notation for his tensor.

This result is called by Riemann the curvature of the n-dimensional space for the given elementary two-dimensional circuit. It is, of course, an invariant and independent of any choice of co-ordinates; it is dependent upon the ratios of the $dS^{\lambda\mu}$, that is, on the orientation of the circuit at the point. If

it so happens that each component of the covariant curvature tensor $(\kappa\lambda\mu\nu)$ or $R_{\lambda\mu\nu\kappa}$ bears the same ratio to the corresponding determinant $|\kappa\lambda, \mu\nu|$ as every other, then the curvature is independent of the orientation and is uniform.

This result is clearly a generalisation of Gauss' result for the curvature of a two-dimensional continuum.

The link between the Einstein Theory of Gravitation and Riemann's geometrical researches is now obvious. Einstein, in adopting a general quadratic expression for the value of the square of the invariant separation between two physical events, was naturally led to use the same mathematical analysis as had been developed for the treatment of a multi-dimensional continuum. In space-time where gravitation is absent, or at most entirely removable, every component of $R_{\lambda\mu\nu}{}^{\kappa}$ or $R_{\lambda\mu}$ is zero, or in geometrical language such a space-time is a flat "homaloidal" four-dimensional continuum. In it any vector carried round any closed circuit by a parallel displacement would return undeviated from its original direction. In actual space-time, except at a great distance from all matter, this is not so; there is a curvature. A vector will not return undeviated; yet the curvature is not arbitrary. There is a certain limitation on it expressed by

$$R_{\lambda\mu} \text{ (or } G_{\lambda\mu}) = 0.$$

Taking equation (15), it is not difficult to interpret this geometrically in the following manner. Suppose we co-ordinate the points of an n-dimensional continuum in any way; consider the point P and neighbouring points $P_1, P_2, \ldots, P_n$, so that P_λ is on the local λ-co-ordinate axis through P, and so that the x_λ-co-ordinate of P_λ differs from that of P by unity, the other co-ordinates of P_λ being, of course, the same as that of P. With P and any pair of these points P_λ and P_μ, we can construct an elementary parallelogram; we shall call it a $\lambda\mu$-parallelogram. Now carry a vector at P originally directed along the λ-axis around the $\mu 1$ parallelogram; on returning to P it will in general be deviated and have a component along the 1-axis, which was of course originally zero. Next carry the vector round the $\mu 2$ parallelogram and obtain a component along the 2-axis. Repeat this operation for all the parallelograms $\mu 1$, $\mu 2$, $\mu 3$, . . ., μn. $R_{\lambda\mu}$ is proportional to the sum of the components thus obtained. So if Einstein's law is true for the continuum, the sum of these components is zero. (Each one of them would, of course, be zero in a homaloidal continuum.)

In space-time, there would only be two components to add, for there are only four parallelograms for each $\lambda\mu$, one of which $\mu\mu$ yields no individual change, while the $\mu\lambda$-parallelogram also gives no individual component along the λ-axis, since $R_{\lambda\mu\lambda}{}^{\lambda}$ (not summed) is identically zero. So that if we carry a vector directed along the 1-axis round the 23-parallelogram, and obtain the 3-component after the change, it is equal in value and of opposite sign to the 4-component of the vector after it has been carried around the 24-parallelogram.

The deviation of a vector may be presented in physical terms of space and time separately thus. Two particles are projected with equal velocities and directions from two neighbouring points. In a non-gravitational field the line joining them remains parallel to itself in the Euclidean sense. In a gravitational field it does not do so in general.

There is also a method of treating the curvature of an n-dimensional continuum which resembles that first used in the early part of this chapter for a surface, and which brings out forcibly the significance of the invariant G or the "scalar of curvature."

Consider the points of the continuum to be curved in a homaloidal $(n + 1)$-dimensional continuum; refer it to axes so chosen that the equation of the n-dimensional continuum is

$$2z = k_1x_1{}^2 + k_2x_2{}^2 + \ldots + k_nx_n{}^2 + \text{higher powers.}$$

This means that the origin is a point of the n-continuum, and the axis of z is normal to it, while the axes of $x_1, x_2, \ldots, x_n$ are tangential to it and parallel to the lines of principal curvature.

Then

$$\begin{aligned}\delta s^2 &= \delta z^2 + \delta x_1{}^2 + \delta x_2{}^2 + \ldots + \delta x_n{}^2\\ &= (1 + k_1{}^2x_1{}^2)\delta x_1{}^2 + \ldots + 2k_1k_2x_1x_2\delta x_1\delta x_2 + \ldots\end{aligned}$$

Hence

$$\begin{aligned}g_{\lambda\lambda} &= 1 + k_\lambda{}^2x_\lambda{}^2\\ g_{\lambda\mu} &= k_\lambda k_\mu x_\lambda x_\mu.\end{aligned}$$

On working out the Christoffel indices and going to the limit at the origin, it can be shown that

$$G = -\sum_\lambda\sum_\mu k_\lambda k_\mu.$$

For space-time

$$G = -2(k_1k_2 + k_1k_3 + k_1k_4 + k_2k_3 + k_2k_4 + k_3k_4).$$

The k_λ are, of course, reciprocals of the principal radii of curvature.

In a two-dimensional continuum curved in three dimensions, $k_1 k_2$ is the Gauss measure of curvature. In space-time curved in five dimensions we have the sum of six similar terms as a measure of the curvature invariant or scalar of curvature.

*

APPENDIX TO CHAPTER XIV

In carrying through the proof of equation (4) of the chapter by the method of Gauss, it will be convenient to introduce new symbols h_{11}, h_{22}, h_{12} to represent the three determinants:

$$\begin{vmatrix} a_{11} & b_{11} & c_{11} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} \qquad \begin{vmatrix} a_{22} & b_{22} & c_{22} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} \qquad \begin{vmatrix} a_{12} & b_{12} & c_{12} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

respectively, and l, m, n to represent the three direction cosines $(b_1c_2 - b_2c_1)/\sqrt{g}$, etc.

Let r be a principal radius of curvature at a point on the surface whose Cartesian co-ordinates are (x, y, z); the co-ordinates of the corresponding centre of curvature are $(x + lr, y + mr, z + nr)$. But if $(x + \delta x, y + \delta y, z + \delta z)$ is an adjacent point on the corresponding line of curvature through (x, y, z), the co-ordinates of the centre are also $(x + \delta x + (l + \delta l)r$, etc.), and so

$$\begin{aligned} \delta x + r\delta l &= 0 \\ \delta y + r\delta m &= 0 \\ \delta z + r\delta n &= 0, \end{aligned}$$

or

$$\left.\begin{aligned} a_1\delta x_1 + a_2\delta x_2 + r\delta l &= 0 \\ b_1\delta x_1 + b_2\delta x_2 + r\delta m &= 0 \\ c_1\delta x_1 + c_2\delta x_2 + r\delta n &= 0 \end{aligned}\right\} \quad . \quad . \quad . \quad (1)$$

Multiplying these by a_1, b_1, c_1 respectively, and adding, we obtain

$$g_{11}\delta x_1 + g_{12}\delta x_2 + r\Sigma(a_1\delta l) = 0.$$

Now

$$\Sigma a_1 l = 0,$$

hence

$$\begin{aligned} \Sigma(a_1\delta l) &= -\Sigma(l\delta a_1) \\ &= -\delta x_1\Sigma(a_{11}l) - \delta x_2\Sigma(a_{12}l) \\ &= -(h_{11}\delta x_1 + h_{12}\delta x_2)/\sqrt{g}. \end{aligned}$$

So writing ρ for $r/\sqrt{g}$ we obtain

$$(h_{11}\rho - g_{11})\delta x_1 + (h_{12}\rho - g_{12})\delta x_2 = 0. \quad . \quad (2)$$

Similarly multiplication of equations (1) by a_2, b_2, c_2 respectively and addition yield

$$(h_{12}\rho - g_{12})\delta x_1 + (h_{22}\rho - g_{22})\delta x_2 = 0. \quad . \quad (3)$$

The elimination of δx_1 and δx_2 from (2) and (3) gives us a quadratic equation in ρ whose roots are $r_1/\sqrt{g}$ and $r_2/\sqrt{g}$, r_1 and r_2 being the two principal radii of curvature at (x_1, x_2). As this quadratic equation is

$$(h_{11}h_{22} - h_{12}^2)\rho^2 - (h_{11}g_{22} + h_{22}g_{11} - 2h_{12}g_{12})\rho + g = 0,$$

it follows that

$$r_1r_2/g = g/(h_{11}h_{22} - h_{12}^2),$$

and thus Gauss' measure of curvature, viz., $1/r_1r_2$ is equal to

$$(h_{11}h_{22} - h_{12}^2)/g^2. \quad . \quad . \quad . \quad (4)$$

To reduce this to the form quoted in (4) of the chapter, we first observe that by the rule for multiplication of determinants,

$$h_{11}h_{22} = \begin{vmatrix} \Sigma a_{11}a_{22} & \Sigma a_1a_{11} & \Sigma a_2a_{11} \\ \Sigma a_1a_{22} & \Sigma a_1^2 & \Sigma a_1a_2 \\ \Sigma a_2a_{22} & \Sigma a_1a_2 & \Sigma a_2^2 \end{vmatrix}$$

By differentiating the relations

$$\Sigma a_1^2 = g_{11}, \ \Sigma a_1a_2 = g_{12}, \ \Sigma a_2^2 = g_{22}$$

with respect to x_1 and x_2, it can be shown that this determinant is equal to

$$\begin{vmatrix} \partial[11, 2]/\partial x_2 - \Sigma a_2\partial a_{11}/\partial x_2 & [22, 1] & [22, 2] \\ [11, 1] & g_{11} & g_{12} \\ [11, 2] & g_{12} & g_{22} \end{vmatrix}$$

In a similar way h_{12}^2 can be expressed as a determinant and shown to be equal to

$$\begin{vmatrix} \partial[12, 2]/\partial x_1 - \Sigma a_2\partial a_{12}/\partial x_1 & [12, 1] & [12, 2] \\ [12, 1] & g_{11} & g_{12} \\ [12, 2] & g_{12} & g_{22} \end{vmatrix}$$

Bearing in mind that $\partial a_{11}/\partial x_2 = \partial a_{12}/\partial x_1$, we see that

$$\begin{aligned} h_{11}h_{22} - h_{12}^2 = g(&\partial[11, 2]\partial x_2 - \partial[12, 2]\partial x_1) \\ &+ (g_{12}[11, 2] - g_{22}[11, 1])[22, 1] \\ &+ (g_{12}[11, 1] - g_{11}[11, 2])[22, 2] \\ &- (g_{12}[12, 2] - g_{22}[12, 1])[12, 1] \\ &- (g_{12}[12, 1] - g_{11}[12, 2])[12, 2]. \end{aligned}$$

Since

$$g^{11} = g_{22}/g,\ g^{22} = g_{11}/g,\ g^{12} = -g_{12}/g,$$

we find that

$$h_{11}h_{22} - h_{12}^2$$

is equal to the product of g and the expression

$$\partial[11, 2]/\partial x_2 - \partial[12, 2]/\partial x_1 + \{12, \alpha\}[12, \alpha] - \{11, \alpha\}[22, \alpha]$$

(summed for $\alpha = 1$ and 2).

But this expression is R_{1212}, and thus the curvature of the surface is

$$R_{1212}/g.$$

CHAPTER XV.

THE CURVATURE OF SPACE-TIME.

IN the early days of the Relativity Theory a number of objections were urged against it which were based on the fallacious assumption that the restricted theory asserted the irrelevance of rotation in an absolute space in the same way as it asserted the irrelevance of uniform motion. Such a statement would clearly have been an unjustifiable interpretation of the postulates of the restricted principle. Indeed, it was known that in so far as mechanical effects are concerned, the invariance of the Newtonian laws for a change to a frame of reference in uniform *translatory* motion relative to the original is not accompanied by a similar invariance for a change to a frame in uniform *rotatory* motion relative to the original. And as far as optical effects are concerned, the experiments of Sagnac have left no doubt of the important fact that rotation cannot be so lightly dismissed from consideration as translation. In Newtonian theory acceleration and rotation have a character apparently more absolute than uniform translation. The generalised theory of Relativity has robbed simple acceleration of this absolute character; the change in frames is signalised by a different set of $g_{\mu\nu}$-potentials, and all gravitational fields are assumed to be relative in character. But rotation still appears to retain something of the absolute about it. Consider, for example, the relative rotation of the stars and the earth. From a purely kinematic standpoint it is, of course, immaterial whether we speak of a rotation of the stars around the earth or of the earth relative to stars. Dynamically, however, we cannot regard the matter as satisfactorily settled if we consider the alternatives equally true in any fundamental sense. We find, to quote the famous experiment of Foucault, that the plane of vibration of a pendulum on the earth's surface rotates slowly around the vertical in a manner which suggests that the plane of vibration is maintaining an unchanged orientation in an absolute space in which the earth is rotating. Experiments with as accurately balanced a gyroscope as can be obtained also suggest that the axes of a perfectly supported gyroscope

would preserve an absolute direction which would, of course, be changing relative to directions marked on the earth's surface. Then there is the phenomenon of the protuberance at the earth's equator, accompanied by the flattening at its poles. Moreover the rate of rotation calculated from experiments with the gyroscope and pendulum is very exactly equal to the rate of relative rotation of the earth and stars as directly observed, so that even were the stars invisible to us we could still detect the earth's rotation relative to something. This would suggest that there is an absolute space, and while uniform motion of translation in the space produces no discernible change in any physical experiment which would permit of a measurement of the speed of such motion to be made, motion of rotation does affect the course of such experiments and thereby yields a measure of its actual magnitude.

The natural reply to this is to point out that the transformation of co-ordinates from one frame of reference to another in rotation relative to it leads to a modification of the potentials which would certainly produce a type of motion in the second frame differing from that which occurs relative to the first. For instance, in the so-called "inertial frame," in which the "fixed" stars are not really fixed, but still have comparatively slow movements, the matter tensor $T_{\mu\nu}$ would proclaim its existence most decidedly by the component T_{44} on account of the smallness of the velocities of matter in the frame compared to the speed of light. By contrast, in a frame turning with the earth, the speed of these same stars would be relatively enormous, and the whole character of the matter tensor would be altered.* These very great velocities, even for matter so distant, might produce, therefore, gravitational effects responsible for the behaviour of Foucault's pendulum and the gyroscope. Indeed, many years before the appearance of the Relativity Theory, Mach, in his Principles of Mechanics, had forsaken the philosophy of an absolute space and attempted to present these occurrences from a standpoint somewhat similar to that just outlined.

But there still remains an objection which may be urged against the relativist. Doubtless it is true that the transformation of co-ordinates involved in introducing a rotating frame of reference makes no difference in the form of the

* The author has had on various occasions to point out to the perplexed that the fact that velocity of matter cannot surpass a certain invariant velocity (that of light) is a deduction of the restricted theory, and must not be transported without qualification into the general.

differential equations of motion, but it does make a decided difference in the limiting values of the $g_{\mu\nu}$ as events are removed further and further from the neighbourhood of matter, i.e., there is a change in the boundary conditions.

In Newtonian theory the values of the potential function are determined not only by Laplace's and Poisson's equations, but also by the condition that they vanish at infinity in any frame of reference. Now it requires no great trouble to see that no similar statement is possible in Einstein's theory as far as it has been developed in the previous chapters. The equations of gravitation are not linear in the $g_{\mu\nu}$, and so the fields of a number of particles are not strictly additive ; yet as any natural gravitational field disturbs the values of the potentials from the values :

$$\left.\begin{matrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 \end{matrix}\right\} \quad . \quad . \quad . \quad (1)$$

by very small amounts, it is not seriously inaccurate to regard the solution for a group of bodies as the sum of solutions for the individual bodies. So that if V is the value of the *Newtonian* potential for a number of bodies in comparatively slow motion in a frame of reference, the values for the $g_{\mu\nu}$ in this frame differ from the following scheme by negligible amounts.

$$\left.\begin{matrix} -(1+2V) & 0 & 0 & 0 \\ 0 & -(1+2V) & 0 & 0 \\ 0 & 0 & -(1+2V) & 0 \\ 0 & 0 & 0 & 1-2V \end{matrix}\right\} \quad . \quad (2)$$

At great distances from all matter it is clear that the scheme (2) approaches the scheme (1).

Suppose now we transform to a frame accelerated with respect to the former, so that we have a transformation such as

$$x_1' = x_1 + ax_4^2,$$

the other co-ordinates remaining unchanged, it is quite easy to verify that the values of $g_{\mu\nu}$ are transformed in such a manner that as x_4 approaches ∞ the values of the $g_{\mu\nu}$ approach the scheme.

$$\left.\begin{matrix} -1 & 0 & 0 & \infty \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \infty^2 \end{matrix}\right\} \quad . \quad . \quad . \quad (3)$$

a quite different set of boundary values to (1). Similarly, if we change to a frame in rotation relative to the first, say, by a transformation such as

$$\phi' = \phi + \omega t,$$

we once more find infinite values appearing in the limits of certain of the $g_{\mu\nu}$.

It is therefore not an unreasonable hypothesis if anyone chose to assert that although we cannot single out any frame of reference as privileged by means of the differential equations, we could conclude that frames in which the boundary conditions correspond to the simple scheme of the restricted theory, viz., (1), have an element of absoluteness about them which is denied to others. As to how closely actual values of the $g_{\mu\nu}$ do approach the scheme (1) in space accessible to observation, there appears to be little doubt. De Sitter asserts that " within the solar system the limits of uncertainty are very narrow: say the eighth decimal place. As we get further away in space, or in time, or in both, the limits become wider: at a distance of a million light years we can perhaps only guarantee the second decimal place. How the $g_{\mu\nu}$ are in those portions of space and time to which our observations have not yet penetrated, we do not know, and how they are at infinity of space or of time we shall never know." * These statements are based on the slight deviation of the spectral lines of even the most distant observed bodies, such as the spiral nebulæ, from the positions for terrestrial sources, and on the smallness of the relative velocities of the stars.

There appears, therefore, at first sight something absolute about the scheme of boundary values given in (1). In Relativity Theory there is strictly no distinction between gravitation and inertia. They are both fused in the $g_{\mu\nu}$, at all events so long as we regard the differential equations alone. But the remarks just made show that the boundary conditions would seem to single out those parts of the $g_{\mu\nu}$ written in (1) or (3) as distinct from the remaining parts, which can be directly traced to the effect of known matter, so that it might be legitimate on that account to refer to the latter parts as " gravitation " and the former as " inertia." But if we transform to other axes, we have seen that while the deviations of the $g_{\mu\nu}$ from the values at infinity can still be traced to known matter and called " gravitation," the " inertia " has quite different values to those occurring in the first frame.

* " Monthly Notices," Nov. 1917.

Now to the thorough-going relativist who asserts that Relativity has a philosophic foundation and justification apart from the teachings of an experimental science, this is decidedly awkward ; but it is not the business of the upholders of the physical principle of Relativity to defend a particular philosophic standpoint. Nevertheless, in the nature of things they must concern themselves with the problem thus raised and consider fairly if further investigation can yield any support to the postulate that there is a " Relativity of inertia," as well as of space, time, electromagnetism, and gravitation.

Now if such a contention is to be supported, very little examination is required to see that the only set of boundary values at infinity for the $g_{\mu\nu}$ which would be invariant for all transformations are

$$\left.\begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}\right\} \quad . \quad . \quad . \quad (4)$$

Thus, the whole of the $g_{\mu\nu}$ would be deviations from the values at infinity, and the " inertia " as well as gravitation would be due to the effects of matter. Such a conclusion would be entirely in agreement with the standpoint of Mach and his followers, who assert that if all matter were destroyed with the exception of one material particle, then this particle would not possess inertia, and, as will appear later, if in supporting this suggestion all matter known to us—stars, nebulæ, clusters, etc.—prove insufficient, they assume the existence of still more matter hitherto unobserved. Before proceeding to investigate the possible validity of the scheme (4), however, the reader must carefully note that the introduction of this scheme involves a postulate. De Sitter calls it the " mathematical postulate of the Relativity of inertia."

We have to consider how it is possible to obtain values of the $g_{\mu\nu}$-potentials which will, in the part of the world accessible to observation, preserve values differing so little from those as obtained hitherto that the experimental tests are still satisfied, and yet at greater distances will gradually approach the scheme (4) or any other scheme which might prove to be an invariant set of boundary values for defined types of transformation.

It appears, on investigation, that such values are obtainable by a slight modification of the law of gravitation, which would become

$$
\left.\begin{aligned}
& G_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}(G - 2\lambda) = -\,8\pi\kappa T_{\mu\nu} \\
\text{or}\quad & G_{\mu\nu} - \lambda g_{\mu\nu} = -\,8\pi\kappa(T_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}T)
\end{aligned}\right\} \quad . \quad . \quad (5)
$$

from which is deduced

$$G - 4\lambda = 8\pi\kappa T. \quad . \quad . \quad . \quad . \quad (6)$$

As will be shown presently, these equations yield values of $g_{\mu\nu}$ which satisfy the suggested boundary conditions at enormous distances, provided the constant λ is chosen so small that the previous law

$$G_{\mu\nu} = -\,8\pi\kappa(T_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}T)$$

is practically obeyed at distances of a smaller order of magnitude. But, in order not to interrupt the more general discussion of the point involved, we shall defer the proof for a little. One very striking conclusion will make its appearance, viz., that the three-dimensional world is not infinite, but is finite in size though unbounded, just as a two-dimensional surface like that of a sphere or ellipsoid is unbounded though of finite dimensions. Indeed, if we consider the integral

$$\int_0^\infty \sqrt{(-\,g_{11})}\,dx_1$$

which is the length of the semi-axis of x_1 in natural measure, it is clearly finite if g_{11} approaches a zero of sufficiently high order as x_1 increases, and it will be seen that in the solutions of (5) g_{11}, g_{22}, g_{33} satisfy this condition.

In his book " Raum, Zeit, Materie," 4th edition, section 34, p. 248 (English translation, p. 273), Weyl does not admit the cogency of these arguments based by Einstein on considerations of the interconnection of the world as a whole and leading to this modification of boundary conditions. He, on the other hand, contends that the differential equations in themselves contain the physical laws of nature in an unabbreviated form, and even without boundary conditions all ambiguity is excluded. Thus, for example, the laws of the electromagnetic and gravitational fields satisfy the principle of causality, i.e., the time derivatives of the " phase quantities " A_λ, $F_{\mu\nu}$, $g_{\mu\nu}$, $\{\lambda\mu,\nu\}$ are expressed in terms of these quantities themselves and their differential coefficients with respect to the space co-ordinates, so that from the state of affairs at one instant we can predict the state of affairs existing at the next. Further, we have no occasion to concern ourselves about boundary conditions at enormous distances beyond our present powers of observation,

for our co-ordination of events is only valid without ambiguity in the neighbourhood of some definite world event. The argument is a subtle one, and may perhaps be made more clear by a consideration of Gaussian co-ordinates on a surface. In general, the lines forming the " meshes " of the Gauss method intersect in more than one point on the surface, and so these meshes can only be used unambiguously to define the co-ordinates of points on a limited position of the surface, and similar limitations must in general hold for any co-ordination of the events of the world. One of the most obvious ways of effecting co-ordination in the world by the Gauss method is to take a definite point-instant of the world as origin O, and introduce co-ordinates x_1, x_2, x_3, x_4, such that at O itself

$$\delta s^2 = \delta x_4^2 - \delta x_1^2 - \delta x_2^2 - \delta x_3^2.$$

There is a three-dimensional space $x_4 = 0$ which surrounds O. In this we mark off a region R such that in it

$$\delta x_1^2 + \delta x_2^2 + \delta x_3^2$$

is everywhere definitely positive. From each point of R draw the geodesic line of the world which is orthogonal to R, and draw it in the time-like direction, i.e., " towards the future." These lines will fill up without intersection, and therefore without ambiguity, a certain four-dimensional neighbourhood of O. Now assign to any event or world-point in this neighbourhood co-ordinates such that x_1, x_2, x_3 have the same values as were assigned to the world-point in R from which started the geodesic passing through the event in question, and x_4 has the value equal to the proper time or separation between the world-point in R and the chosen world-point. Taking a definite value for x_4, this will define for us a region R′ of another three-dimensional space surrounding the point (0, 0, 0, x_4), and it is not difficult to show that the geodesics are orthogonal to this region also. Thus along any of the geodesics in question there is no change in x_1, x_2, x_3, and so

$$ds^2 = dx_4^2$$

or

$$g_{44} = 1.$$

The geodesics are also orthogonal to $x_4 = 0$. Hence

$$g_{14} = g_{24} = g_{34} = 0^* \quad . \quad . \quad . \quad (7)$$

* This can be seen by an appeal to three-dimensional Euclidean geometry. If the axis of z is normal to the plane of xy, but the axes of x and y are not necessarily at right angles, the square of an element of length might contain a term in $\delta x \delta y$, but not terms in $\delta x \delta z$ or $\delta y \delta z$.

provided $\dot{x}_4 = 0$. Since, along a geodesic, x_1, x_2, x_3 do not vary, it can be seen from the equations of a geodesic

$$\ddot{x}_\mu + \{\alpha\beta, \mu\}\dot{x}_\alpha\dot{x}_\beta = 0$$

that

$$\{44, \mu\} = 0$$

if

$$\mu = 1, 2, 3, \text{ or } 4.$$

Hence

$$[44, \mu] = 0,$$

and since $g_{44} = 1$,

$$\partial g_{14}/\partial x_4 = \partial g_{24}/\partial x_4 = \partial g_{34}/\partial x_4 = 0. \quad . \quad . \quad (8)$$

Combining (7) and (8) we find that

$$g_{14} = g_{24} = g_{34} = 0$$

at the point on a geodesic adjacent to the point in R, i.e., at all points in R′. Thus we arrive at a co-ordination which involves a family of geodesic lines with time-like direction and a one-parameter family of three-dimensional spaces, orthogonal to the lines, viz., $x_4 =$ constant. Using this system of co-ordinates,

$$\{\mu\nu, 4\} = -\tfrac{1}{2}\partial g_{\mu\nu}/\partial x_4 \quad (\mu, \nu = 1, 2, \text{ or } 3),$$

and so by means of the gravitational equations we can express the derivatives

$$\partial\{\mu\nu, 4\}/\partial x_4$$

in terms of the $g_{\mu\nu}$, their first and second order derivatives *with respect to* x_1, x_2, x_3 and the $\{\mu\nu, 4\}$ themselves. So the principle of causality is satisfied, but we cannot say how far we may proceed along the geodesics or travel out into the family of three-dimensional regions until intersection takes place and it becomes impossible to express the co-ordinates of all events in this manner in a continuous and singly-reversible (" umkehrbar eindeutig ") way.

But although Weyl questions the validity of Einstein's argument from boundary conditions, he does not, in consequence, oppose the conclusions, but he attributes greater weight to the following consideration, which would also suggest the same result.

In our stellar system the relative velocities of the stars are remarkably small compared to the speed of light. How does

such a system persist? How is it that it has not in ages past dispersed into infinite space? To a Maxwell "demon" the problem of the persistence of an enclosed gas system would present itself in a somewhat similar manner. Yet we demonstrate the validity of such persistence by statistical methods which also prove that serious departure from uniformity of density is of very brief duration and of very limited extent, and that velocities differing largely from a certain mean are comparatively small in number. But any attempt to apply such an argument to the stellar system breaks down if we apply it in connection with the law of gravitation as hitherto formulated. For an ideal state of equilibrium in which there is a uniform distribution of bodies at rest in a static gravitational field is irreconcilable with such laws. On the Newtonian theory such a distribution in infinite space would lead to an indefinite and infinite value for the intensity of gravitational force at any defined point. On Einstein's theory the world-line of a particle at rest, i.e., a line along which x_1, x_2, x_3 are constant, is a geodesic if

$$\{44, \mu\} = 0 \qquad (\mu = 1, 2, \text{or } 3),$$

or

$$[44, \mu] = 0,$$

and therefore

$$\partial g_{44}/\partial x_\mu = 0.$$

Thus such a distribution at rest is only possible if

$$g_{44} = \text{constant},$$

which is incompatible with the Einstein law of gravitation, as can be seen readily by remembering that in the first approximation $1 - g_{44}$ is equal to twice the Newtonian potential, and so

$$\Delta g_{44} = 4\pi\kappa\mu.$$

That being so, it is pointed out by Weyl that, if we appeal to the Principle of Stationary Action, it is clear we could have introduced just as readily for the action of the gravitation field the invariant integral

$$\int q(aG + \beta)d\omega$$

instead of

$$\int qGd\omega$$

where a and β are numerical constants. This is clearly equivalent to (6) above, and it appears that the ideal state of equilibrium considered above is compatible with this modification.

The demonstration of this we shall also defer for the present, and deal with it along with the finiteness of space at a later stage, for there is a third line of argument due to Einstein distinct from his first line, and from Weyl's, which leads us once more to the same modification of the law of gravitation as before.

At present Physical Science is committed to an electrical theory of matter. All matter is constituted of charged corpuscles (electrons, nuclei). The inertia of matter is but a measure of electromagnetic energy, either "bound" to these corpuscles or "free," in the form of radiant energy. One essential difference between the "bound" and "free" energies is the fact that the proper or rest mass of the former is not zero, while it is so for the latter. This may be seen in a simple manner by the aid of the restricted theory; for the mass of anything travelling with the speed of light would be infinite unless its rest mass were zero. But this is a somewhat imperfect manner of stating the distinction. From the point of view of the generalised theory, the scalar T of the matter-tensor $T_{\mu\nu}$ is not zero, but equal to μ_0 the proper density of the matter, while the scalar E of the electromagnetic energy-momentum tensor $E_{\mu\nu}$ is zero, as can be proved easily by a reference to the expression for it at the end of Chapter X. The origin of this distinction is closely connected with a very serious difficulty in the electrical theory of matter, which still awaits a solution. This difficulty is simply the existence of the electron itself. If we accept it as a fundamental concept—as the chemists of the nineteenth century accepted the atom—the difficulty does not arise; but if we wish to treat it as a structure and derive it from the equations of the field, we are at once perplexed by the fact that it coheres and does not dissipate itself under the mutual repulsion of its parts. Some years ago Poincaré suggested that the non-Maxwellian cohesive forces might be regarded as a *pressure* whose value could be represented by

$$2\pi\sigma^2$$

where σ would stand for the surface density of charge on the electron; this expression being derived from the well-known electrostatic theorem that the repulsion on a part of the surface charge of a conductor can be represented as a normal tension on the surface of amount $2\pi\sigma^2$ per unit of area. Such cohesive forces would, of course, introduce a new term into the energy of the field, whose amount would be equal to the product of

the average pressure and the volume of the electron. Conceiving, for illustrative purposes alone, the form of the electron to be spherical, the additional energy would be

$$\tfrac{4}{3}\pi a^3 2\pi\sigma^2,$$

which is equal to

$$8\pi^2 a^3(e/4\pi a^2)^2/3$$
$$= e^2/6a$$

where e is the electronic charge and a the radius. Now the electrostatic energy of the field of a spherical electron at rest is, of course, by elementary theory

$$e^2/2a.$$

Hence the energy of field and cohesive forces is $2e^2/3a$, which is actually the well-known expression for the rest energy (or mass) of a spherical electron derived by treating the magnetic energy due to its motion as kinetic energy.

Of course, the existence of such a pressure as Poincaré suggests could not be deduced from the present theory of the electromagnetic field, but bearing in mind the nature of the Relativity Principle as a test or criterion for physical theories, we must be prepared to consider whether any tentative hypothesis can pass this test. The gravitational equation first suggested by Einstein was

$$G_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}G = -8\pi\kappa T_{\mu\nu}.$$

$T_{\mu\nu}$ is the matter tensor and no theory as to the nature of matter is involved. This equation is of the kind called "macroscopic," i.e., it presupposes that the granular structure of matter has been smoothed out, and the density occurring in $T_{\mu\nu}$ is an averaged density. If, however, we wish to regard matter "microscopically" as a collection of charged particles, we must replace the right-hand side by

$$-8\pi\kappa E_{\mu\nu},$$

for this gives expression to the hypothesis that matter is electrical in origin. But the scalar invariant E of $E_{\mu\nu}$ is zero, and this introduces a contradiction, for the scalar of the left-hand side is $-G$, which cannot be zero within matter.

This contradiction can be removed by choosing in the microscopic equation a tensor involving $G_{\mu\nu}$ and $g_{\mu\nu}$ whose invariant is zero, and it is easy to see that such a condition is satisfied by the equation

$$G_{\mu\nu} - \tfrac{1}{4}g_{\mu\nu}G = -8\pi\kappa E_{\mu\nu}. \quad . \quad . \quad (9)$$

This leads to the mixed tensor equation

$$G_\mu^\nu - \tfrac{1}{4}g_\mu^\nu G = -8\pi\kappa E_\mu^\nu. \qquad . \qquad . \qquad (10)$$

But as the contracted derivative

$$(G_\mu^a)_a - \tfrac{1}{2}(g_\mu^a G)_a$$

is identically zero, it follows from (10) that

$$\tfrac{1}{4}(g_\mu^a G)_a = -8\pi\kappa(E_\mu^a)_a$$

or

$$\tfrac{1}{4}\partial G/\partial x_a = 8\pi\kappa F_{\mu a}J^a. \qquad . \qquad . \qquad . \qquad (11)$$

Therefore in the "void," i.e., outside the world filaments of the electrons, but including the world tubes of any spatial regions containing "free" energy of radiation, the curvature G is constant, because since J^μ is zero in such regions of the world,

$$\partial G/\partial x_\mu = 0.$$

As G is the same in all parts of the void, whether they contain free energy or not, we see that this constant value of G, represented by G_0, satisfies

$$G_{\mu\nu} - \tfrac{1}{4}g_{\mu\nu}G = 0$$

or the new equation for gravitation outside matter is

$$G_{\mu\nu} - \lambda g_{\mu\nu} = 0$$

where

$$\lambda = G_0/4.$$

This suggests that the new *macroscopic* equation within matter is

$$\left.\begin{aligned} G_{\mu\nu}' - \tfrac{1}{2}g_{\mu\nu}G' &= -8\pi\kappa T_{\mu\nu} \\ G_{\mu\nu}' &= -8\pi\kappa(T_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}T) \end{aligned}\right\} \qquad . \qquad . \qquad (12)$$

or

where we write $G_{\mu\nu}'$ for the tensor $G_{\mu\nu} - \lambda g_{\mu\nu}$, and therefore G' for $G - 4\lambda$.

By contraction

$$8\pi\kappa T = G'$$

and by contracted derivation

$$-8\pi\kappa(T_\mu^a)_a$$

$$\begin{aligned}
&= \text{the contracted derivative of } G'{}_\mu^\nu - \tfrac{1}{2}g_\mu^\nu G' \\
&= \text{,,} \qquad \text{,,} \qquad \text{,,} \qquad \text{,,}\ G_\mu^\nu - \lambda g_\mu^\nu - \tfrac{1}{2}g_\mu^\nu(G - 4\lambda) \\
&= \text{,,} \qquad \text{,,} \qquad \text{,,} \qquad \text{,,}\ G_\mu^\nu - \tfrac{1}{2}g_\mu^\nu G + \lambda g_\mu^\nu \\
&= 0 + \lambda(g_\mu^a)_a \\
&= 0
\end{aligned}$$

and so the laws of conservation of energy and momentum are still satisfied.

In this G is an *average* curvature for a " physically small " extension of the world, and we see that this average scalar curvature exceeds the scalar curvature in the void by an amount depending on the amount of matter present. We can, if we please return to the microscopic aspect by considering equation (11). If dx_μ/ds refers to derivation along the world-line of an electron,

$$\begin{aligned}\tfrac{1}{4}\partial G/\partial x_a \,.\, dx_a/ds &= -\,8\pi\kappa F_{a\beta}J^\beta dx_a/ds \\ &= -\,8\pi\kappa\rho_0{}^{-1}F_{a\beta}J^aJ^\beta\end{aligned}$$

where ρ_0 is the proper charge density of the electron.

But since $F_{\mu\nu}$ is anti-symmetric, the right-hand side is zero.

Therefore $$dG/ds = 0,$$

or G is constant along the world-line of an electron.

This gives us the picture of extremely thin world-filaments within which there is a constant curvature G_f; between the filaments lies a portion of the void where there is another and smaller constant curvature G_0. This is the microscopic view. Macroscopically, filaments and void make up the world-tubes of " matter " throughout which there exists an average curvature G, smaller than G_f but larger than G_0. The remaining parts of the void are " outside matter," but in them the curvature is still G_0, and not zero, as in the equation first proposed by Einstein.

From the macroscopic point of view $G - G_0$ determines the density of matter ; from the microscopic, $G_f - G_0$ determines the non-Maxwellian binding forces of the electron or the Poincaré pressure.

Along all these different avenues of approach the evidence brings home to us the necessity for a modification of the law of gravitation to the form suggested above in (5), viz.,

$$G_{\mu\nu} - \lambda g_{\mu\nu} = -\,8\pi\kappa(T_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}T)$$

leading to the scalar equation

$$G - 4\lambda = 8\pi\kappa T,$$

where λ is a constant curvature which may, if we please, be taken so small that the modification does not interfere with the experimental observations.

Addressing ourselves now to the problem of finding an

expression for the separation element and a distribution of matter which will satisfy this equation, we make the following simplifying assumptions, viz., neglect all pressures and internal forces and suppose all matter to be at rest.

The covariant matter tensor $T_{\mu\nu}$ is given by

$$\begin{aligned} T_{\mu\nu} &= g_{\mu\alpha}g_{\nu\beta}T^{\alpha\beta} \\ &= g_{\mu\alpha}g_{\nu\beta}\mu_0\dot{x}_\alpha\dot{x}_\beta \end{aligned}$$

where μ_0 is the proper density of matter.

With the assumptions just made all the $T_{\mu\nu}$ are zero, except T_{44}, which is equal to

$$\begin{aligned} & g_{4\alpha}g_{4\beta}\mu_0\dot{x}_\alpha\dot{x}_\beta \\ &= g_{44}g_{44}\mu_0\dot{x}_4^{\,2} \\ &= g_{44}\mu_0, \end{aligned}$$

since

$$\delta s^2 = g_{44}\delta x_4^{\,2},$$

by reason the statical nature of the distribution.

Now put

$$\mu_0 = \mu_a + \mu_g$$

where μ_a is the average proper density and μ_g represents local deviations from this average which are responsible for purely gravitational disturbances. Neglecting these local disturbances, i.e., supposing $\mu_g = 0$, we have from (5)

$$\left.\begin{aligned} G_{11} - \lambda g_{11} &= 4\pi\kappa\mu_a g_{11} \\ &\text{two similar equations in 22 and 33} \\ G_{44} - \lambda g_{44} &= -\,8\pi\kappa\mu_a g_{44} + 4\pi\kappa\mu_a g_{44} \\ &= -\,4\pi\kappa\mu_a g_{44} \end{aligned}\right\} \quad . \quad (13)$$

Einstein suggests that on the vast scale which we are considering (still neglecting local disturbances of the $g_{\mu\nu}$ due to local deviations of the matter density from μ_a) the following expression for δs^2 holds good :

$$\delta s^2 = \delta x_4^{\,2} - \delta x_1^{\,2} - R^2 \sin^2 (x_1/R)(\delta x_2^{\,2} + \sin^2 x_2 \delta x_3^{\,2}) . \quad (14)$$

where R is a constant length of enormous value.

The values of $g_{\mu\nu}$ are

$$g_{11} = -\,1, g_{22} = -\,R^2 \sin^2 (x_1/R), g_{33} = -\,R^2 \sin^2 (x_1/R) \sin^2 x_2, \quad g_{44} = 1$$

and from these it can be calculated that

$$G_{11} = 2g_{11}/R^2,\ G_{22} = 2g_{22}/R^2,\ G_{33} = 2g_{33}/R^2,\ G_{44} = 0,$$

which agree with (13), provided

$$\lambda = 1/R^2 = 4\pi\kappa\mu_a. \qquad (15)$$

The solution (14) corresponds to an expression for a line element $\delta\sigma$ *in space* given by

$$\delta\sigma^2 = \delta r^2 + R^2 \sin^2 (r/R)(\delta\theta^2 + \sin^2 \theta\delta\phi^2) \qquad (16)$$

where we regard x_1, x_2, x_3 as polar co-ordinates r, θ, ϕ.

Expression (16), however, differs from the customary expression, viz.,

$$\delta\sigma^2 = \delta r^2 + r^2(\delta\theta^2 + \sin^2 \theta\delta\phi^2) \qquad (17)$$

in such a way that it implies that our three-dimensional universe is the unbounded but finite hypersurface of a four-dimensional hypersphere of radius R. Near enough to the origin, sin (r/R) can be confounded with r/R and (16) and (17) agree. As we proceed further from the origin, however, actual spheres drawn in our space with radius r begin to behave like small circles on a sphere drawn round a given point as centre, with increasing spherical radius. Just as the circumference of the latter is not $2\pi r$ but $2\pi R \sin (r/R)$, so the area of the surface of a sphere becomes on the hypothesis in (16)

$$4\pi R^2 \sin^2 (r/R).$$

This increases to a maximum as r increases to $\pi R/2$, and thereafter decreases to zero for the limiting distance $r = \pi R$.

The volume of our universe is then finite and given by

$$\int_0^{\pi R} 4\pi R^2 \sin^2 (r/R)dr, \text{ or } 2\pi^2R^3.$$

As the average density of the matter of the world is given by (15), we find for the total mass of the universe

$$\pi R/2\kappa.$$

Thus the size of the universe depends on its mass.

If we follow out the consequences of this hypothesis of Einstein we arrive at some bizarre conclusions. They can be obtained most readily by keeping in one's mind the analogy of a spherical two-dimensional continuum. Taking two points A and B, and then considering a third point C gradually receding from them, at first $\angle$CAB + $\angle$CBA is less than two right angles, and so anyone who assumes that space is " flat " finds that $\angle$ACB is positive, and there is a positive parallax. As

the distance of C from AB increases to $\pi R/2$, and beyond that to πR, $\angle$CAB + $\angle$CBA increases to two right angles, and beyond that to three, so that the parallax of C would by inference become zero and then negative. Rays of light emitted from a point and following the great circle tracks of this space would return to a focus at the antipodal point, and diverging once more would produce the appearance of suns and stars which would be merely "ghosts." The fascinating picture is, however, somewhat spoiled by the thought that local gravitational disturbances would probably deflect the rays, and this aberration would at least destroy the perfection of focus; not to mention that absorption by the "world-matter" would probably reduce the energy of the beams to zero ere the journey round the universe could be completed. However, if, for example, an "anti-sun" did exist, it would appear not at the antipodal point of the sun as it is now situated, but as it was situated in ages past when the light now reaching us was emitted. Indeed, as several journeys round the universe could be assumed on the basis of such speculation, there might be several ghosts of every star haunting regions formerly occupied by it, or regions antipodal to these. One thing which the expression $\pi R/2\kappa$ makes quite clear is the inadequacy of the amount of matter known to us to fit this new hypothesis of Einstein. Thus the mass of the sun, when multiplied by the astronomical constant κ, has the dimensions of a length whose magnitude is 150,000 cms., or 1·5 kilometres.* (See Chapter XIII., p. 280.) It is estimated that our stellar system has a mass of the order 10^9 that of the sun. On the supposition that the spiral nebulæ represent 10^6 such systems, we find that the R corresponding to this mass is about 10^{15} kilometres or 30 parsecs, which is not even as large as the average distance of the stars visible to the naked eye. In order, therefore, to allow for the very large stellar distances known to us, and, further, to permit us to regard them as only a part of the greatest possible distance πR, we must postulate a relatively enormous R, and in consequence the existence of immense quantities of world matter, perhaps diffused through our universe with very small density, but concentrated here and there into stellar systems or existing in innumerable dark stars. Known matter, for example, is insuffi-

* In fact, it is quite customary in works on Relativity to refer to the sun's gravitational mass as 1·5 "kilometres." This, of course, arises from the fact that $\kappa M/r$ is a pure ratio, viz., the ratio of the squared velocity of a particle describing an orbit of radius r to the squared velocity of light.

cient to produce the centrifugal gravitational effects at our earth's surface by its diurnal rotation in our natural frame of reference, and so Einstein's hypothesis is directly in line with the demands of the philosophy of Mach.

Despite the attractiveness of this speculation, it suffers from one serious drawback, which will be evident if we return to the point at which this chapter began, viz., consideration of boundary conditions. As we have written the expression for δs^2 in (14), the question of boundary conditions does not arise, for there is no boundary. But, of course, it is easy to make a transformation of co-ordinates which will permit no limits to be assigned to one of the co-ordinates at least. For example, if we were considering the geometry of the surface of an ordinary sphere, we could make a representation of it on a tangent plane at the origin by writing

$$\varpi = R \tan (r/R),$$

which would involve infinite values of ϖ.

If we make the same transformation in (14), written as

$$\delta s^2 = \delta t^2 - \delta r^2 - R^2 \sin^2 (r/R)(\delta\theta^2 + \sin^2 \theta\delta\phi^2),$$

we obtain

$$\delta s^2 = \delta t^2 - \delta\varpi^2/(1 + \lambda\varpi^2)^2 - \varpi^2(\delta\theta^2 + \sin^2 \theta\delta\phi^2)/(1 + \lambda\varpi^2) \quad (18)$$

where

$$\lambda = 1/R^2.$$

This corresponds to the " flat " world of an observer at the origin, who naturally takes ϖ, θ, ϕ as polar co-ordinates. Of course, (18) shows that this flat world is not governed by Euclidean axioms. For one thing, its scalar curvature is not zero, but is, of course, equal to the finite value for the form (14) from which the transformation has been effected. We make another transformation which gives us Cartesian co-ordinates in this observer's flat world, viz.,

$$\begin{aligned} x_1 &= \varpi \sin \theta \cos \phi \\ x_2 &= \varpi \sin \theta \sin \phi \\ x_3 &= \varpi \cos \theta \\ x_4 &= t, \end{aligned}$$

whence we obtain

$$\delta s^2 = g_{\alpha\beta}\delta x_\alpha \delta x_\beta,$$

where the $g_{\mu\nu}$ are given by the scheme

$$\left.\begin{matrix} -k+\lambda k^2x_1^{\,2} & \lambda k^2x_1x_2 & \lambda k^2x_1x_3 & 0 \\ \lambda k^2x_2x_1 & -k+\lambda k^2x_2^{\,2} & \lambda k^2x_2x_3 & 0 \\ \lambda k^2x_3x_1 & \lambda k^2x_3x_2 & -k+\lambda k^2x_3^{\,2} & 0 \\ 0 & 0 & 0 & 1 \end{matrix}\right\} \quad (19)$$

(We write k for $(1+\lambda\varpi^2)^{-1}$.)

If $\varpi = 0$, (19) degenerates into the scheme (1).

As ϖ increases indefinitely, scheme (19) gradually approaches

$$\left.\begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}\right\} \quad . \quad . \quad (20)$$

Now this is not the scheme (4), which is the only boundary scheme which is invariant for *all* transformations. Scheme (20) is, however, invariant for transformations to accented co-ordinates in which the transformation equation

$$x_4' = f(x_1, x_2, x_3, x_4)$$

degenerates into

$$x_4' = x_4$$

at infinity. Now such a limitation is a violation of the restricted principle of Relativity. As Eddington says: "We have a feeling that Einstein's new hypothesis throws away the substance for the shadow;" * he "has restored the differentiation between space and time by assuming the space-time world to be cylindrical, so that the linear direction gives an absolute time." Or, in the words of de Sitter: "It thus appears that the system only satisfies the mathematical postulate of Relativity if the latter is applied to three-dimensional space alone. In other words, if we conceive the three-dimensional space (x_1, x_2, x_3) with its world matter as movable in an absolute space, its movements can never be detected by observations: all motions of material bodies are relative to the space (x_1, x_2, x_3) with the world matter, not to the absolute space. The world matter thus takes the place of the absolute space in Newton's theory, or of the inertial system. It is nothing else but the inertial system materialised. It should be pointed out that this Relativity of inertia is only realised by making the time practically absolute. It is true that the fundamental equations of the theory, the field equations, and the equations of motion . . . remain invariant for all transformations. But only such

* *Report*, p. 87.

transformations for which at infinity $t' = t$ can be carried out without altering the values (1A)," [our (20)].*

This drawback to the hypothesis of a finite universe is avoided, however, if we accept a suggestion made by de Sitter that on the vast scale considered the element of separation is given by .

$$\delta s^2 = \cos^2 (x_1/R)\delta x_4{}^2 - \delta x_1{}^2 - R^2 \sin^2 (x_1/R)(\delta x_2{}^2 + \sin^2 x_2 \delta x_3{}^2). \quad (21)$$

On working out the $G_{\mu\nu}$ for these values we find that without exception

$$G_{\mu\nu} = 3g_{\mu\nu}/R^2,$$

which satisfies (13)

if $$\lambda = 3/R^2$$

and $$\mu_a = 0.$$

The difference between this hypothesis and that of Einstein is most marked. In Einstein's speculation the space-time world has a natural curvature of 4λ or $4/R^2$ in the void, which, however, is due to the world matter. This curvature of course is greater than this within matter ($G - 4\lambda = 8\pi\kappa T$) ; it becomes 6λ or $6/R^2$, for instance, within matter of average density μ_a.

On de Sitter's assumption there is no world matter ($\mu_a = 0$) ; there are just local " islands " in a vast void. These will, of course, modify locally the natural curvature $3/R^2$ of the space-time world, which is itself due entirely to the modified law of gravitation.

If we write (21) in polar co-ordinates r, θ, ϕ, then transform as before to a flat world tangent at the origin by the transformation

$$\varpi = R \tan (r/R)$$

and then introduce Cartesian co-ordinates, we obtain a scheme of $g_{\mu\nu}$ which agrees with (19), except that the bottom right-hand constituent becomes $(1 + \lambda r^2)^{-1}$ instead of unity. So at infinity we get the limiting scheme (4), which is invariant for all transformations. De Sitter's hypothesis, therefore, involves no breach of the principle of Relativity.

There is one interesting feature of de Sitter's suggestion which holds out some hope of ultimately receiving confirmation. The multiplication of δt^2 in (21) by $\cos^2 (r/R)$ implies that as r increases, the observed time of natural processes increases. The assumption is the same as that raised in the discussion of spectral line displacement in a gravitational field. If some

* "M.N.R.A.S.," LXVIII., p. 3, 1917.

atomic process, for instance, has a universally constant value for its *proper* time, δs, then obviously as r increases and $\cos(r/R)$ diminishes, δt will increase. Indeed, at a distance $\pi R/2$ from the place of observation, δt would be infinite, and all such processes would stop *relative to these observers*. Of course, to observers situate at this place itself, phenomena in the neighbourhood would present their normal appearance, while stagnation would prevail *for them* at our former origin. At the zone $\pi R/2$ away from us light is also reduced to rest; it cannot get round the universe as in Einstein's universe, so there will be no anti-suns; no information can come to us from regions beyond for the same reason. Stars at this zone will have the same parallax as a star at distance R in Euclidean space (this can be seen by consideration of a two-dimensional spherical space), and as stars beyond this barrier will be invisible to us, this parallax will be the least possible.

On account of the slowing down of time at great distances, spectral lines from a source at great distance ought to be displaced somewhat towards the red end as compared with terrestrial sources. Now there is some evidence that there is such an effect in the case of the spiral nebulæ. Of course, velocities of approach or recession of these systems would produce spectral displacement towards violet or red, as the case may be; but if, as seems probable on grounds which have been investigated closely by de Sitter in the paper quoted earlier, velocities of recession should not occur more frequently than velocities of approach, then it would appear that the existence of abnormal Döppler effects (preponderating somewhat towards the red) in the case of objects at enormous distances as compared with our nearer stellar neighbours, yields some little confirmation to de Sitter's world. But the observations are still too few to afford certitude to any statement.

By transformation to other co-ordinates the complete pseudo-sphericity of de Sitter's world can be shown. Thus if we write

$$\sin(r/R) = \sin\zeta \sin\omega$$
$$\tan(\iota t/R) = \cos\zeta \tan\omega$$

we obtain

$$\delta s^2 = -R^2\{\delta\omega^2 + \sin^2\omega[\delta\zeta^2 + \sin^2\zeta(\delta\theta^2 + \sin^2\theta\delta\phi^2)]\},$$

which suggests polar co-ordinates, R, ω, ζ, θ, ϕ, in a five-dimensional space. But as the angles ζ, ω are imaginary, the sphericity is rather spurious, resembling the pseudo-Euclideanism of the space-time world of Minkowski in the restricted theory

after the introduction of imaginary time. If we introduce real angles and write

$$\zeta = \iota\zeta', \ \omega = \iota\omega',$$

we see the hyperbolical form of our world, since in that case

$$\delta s^2 = R^2\{\delta\omega'^2 - \sinh^2\omega'[\delta\zeta'^2 + \sinh^2\zeta'(\delta\theta^2 + \sin^2\theta\delta\phi^2)]\}.$$

In fact, if we write

$$\begin{aligned} x_1 &= R\cosh\omega' \\ x_2 &= R\sinh\omega'\cosh\zeta' \\ x_3 &= R\sinh\omega'\sinh\zeta'\cos\theta \\ x_4 &= R\sinh\omega'\sinh\zeta'\sin\theta\cos\phi \\ x_5 &= R\sinh\omega'\sinh\zeta'\sin\theta\sin\phi \end{aligned}$$

we have

$$x_1^2 - x_2^2 + x_3^2 + x_4^2 + x_5^2 = R^2,$$

which is the equation of a four-dimensional hyperboloid in a space of five dimensions. Also,

$$\delta s^2 = \delta x_2^2 - \delta x_1^2 - \delta x_3^2 - \delta x_4^2 - \delta x_5^2.$$

For de Sitter, space-time has an existence and a form independent of the matter it contains (which only intervenes locally to modify the general curvature); space-time would still exist even if matter were destroyed. Such a conclusion will, of course, be resisted violently by the materialist philosophy of Mach and his followers.

Taking the equations of motion as formulated in Chapter IX. we can obtain the motion of a particle either in Einstein's universe or in de Sitter's, and deduce some interesting conclusions. They are worked out very fully by de Sitter in the paper mentioned above, and the reader is referred to it for further information.

It should be pointed out for the benefit of those unfamiliar with multi-dimensional geometry, that the spherical three-dimensional space of Riemann is not the only kind of finite space which can be conceived to be a physical reality. In Riemann's space all straight lines starting from a point intersect in another point whose distance from the first measured along any of these lines is πR. Many years ago, however, the astronomer Newcomb pointed out that it was possible to lay down a set of axioms and postulates which lead without any contradiction to the conception of a space in which a straight line returns to itself after going round the universe, but does not intersect any line more than once. Newcomb's investigations, which are very

interesting, can be found very clearly and concisely expressed in quite a brief paper to Crelle's "Journal," 83, pp. 293-299 (1877). In the opinion of Einstein and de Sitter, Newcomb's "elliptical" space is really simpler than Riemann's spherical for representation of the finite physical world. In the "elliptical" space there are infinitely many points at the greatest possible distance from a given point, and they lie on a straight line, which is $\frac{1}{2}\pi R$ from the given point. This is in contrast to spherical space, where there is only one point, the "antipodal," which is at the greatest distance, that distance being πR. Right round Newcomb's space is πR, but $2\pi R$ round Riemann's. The volume of the latter is $2\pi^2 R^3$, of the former $\pi^2 R^3$. Of course, in elliptical space, even with Einstein's hypothesis of "cylindrical time," the existence of anti-suns, etc., would no longer follow.

In conclusion, de Sitter has endeavoured to make a rough estimate of the value of R from a variety of astronomical data, and finds values of such an order of magnitude as 10^{12} to 10^{13} radii of the earth's orbit, i.e., about 10^7 parsecs, or 10^{25} cms. This would make the curvature λ about 10^{-50} (cm.)$^{-2}$.

CHAPTER XVI.

Geometry and Physical Forces.

It has been pointed out by more than one writer on Relativity how progress in physical theory during the past quarter-century has been effected by the removal of cramping restrictions on our ideas. The partial abandonment of the concept of a rigid body implied in the Lorentz-Fitzgerald contraction gave an impetus to research which led directly to the earliest formulation of Relativity. The total abandonment of that idea, combined with a freer attitude towards simultaneity, gave us the Minkowski world. The recognition that a more general metric than that used in Euclid's geometry was needed for this world gave us gravitation. It now remains to indicate how a further loosening of bonds can give us electro-magnetic force, and may yet succeed in revealing to us the cohesive forces on the electron which still lie outside Maxwell's electromagnetic theory. Doubtless it is true that the whole theory of the electromagnetic field is perfectly consistent with the hypothesis of general Relativity, but there is not apparently that close connection between the electromagnetic potentials and field tensors and the quantities $g_{\mu\nu}$ which define the metric of the world as exists, according to Einstein, between these same quantities and the gravitational field. It was Weyl * who first pointed out how a still more complete metrical treatment of the world than that used in Riemannian geometry affords a means of identifying the electromagnetic potentials with certain quantities introduced in this wider view of geometry. Eddington † has shown that even Weyl's geometry suffers from a restriction; he has worked out certain conclusions which can be drawn from its abandonment, and suggests that we may thereby find some unsuspected relation between this extremely wide geometry and the non-Maxwellian binding forces on the electron, although he has not succeeded in doing so at present,

* "Sitz. Akad. Preuss.," May, 1918, and his book, "Raum, Zeit, Materie."

† "Proc. R.S." **99**, A. 697, May, 1921. "Phil. Mag.," **42**, Nov. 1921; **43**, Jan. 1922.

and, indeed, makes no assertion that his generalisation will provide the materials for the solution of that difficult problem.

It is a commonplace in the literature of Relativity to point out the nature of our experimental observations. They are almost entirely judgments of coincidences between fiducial marks on instruments such as rulers, protractors, micrometer screws, galvanometer scales, spider lines in eye-pieces, and observed marks on bodies, photographic plates, or optical images. We note that a particular event in one chain of observation coincides in space and time with a particular event in another chain, or that the world-lines of two distinct objects intersect in a certain common world-point. In order to discover relations between such observations we introduce co-ordinates or number-sets, which on our present views may be perfectly arbitrary, provided they are unambiguous and definitely distinguish one event from another, and then set up an expression for a measured interval or separation between two events. This almost amounts to an admission that Bergson is right in asserting that intellectually we do not appreciate the flow of life, but merely visualise static sections or kinematographic pictures of life and reason about these.

One essential feature of this method of treatment is the introduction of the idea of congruency which grew up, of course, with the idea of a rigid body, permanent in configuration. If two marks on one body coincide with two on another, they will always, and at all places, do so provided we take certain well-defined precautions. That is an assertion so eminently reasonable and " commonsensical " that denial of it appears to shake the whole fabric of thought. The bewildered " man in the street " asks in helpless amazement, " What next ! " Yet it is just that statement which Weyl proposes to remove from the category of completely valid assertions, and, as he can easily establish, he defies no definite experimental evidence in so doing. After all, our ideas of a rigid body have already received some rude shocks. In truth, the effects of temperature and pressure on our most " solid " of materials are such that all we could assert of old was that with very definite care we have experience of bodies which as accurately as we can observe, do comply with the idea of congruency, and preserve an unaltered configuration with lapse of time or after movement in space. But the Relativity implied in that statement is apparent to anyone who reflects that provided all bodies change their configuration similarly after experiencing the same displacement, our notions concerning congruency would

require no modification as far as the phenomena were concerned. When Lorentz suggested that movement might contract the "rigid" body, the conditions for "congruent figures" had to be closely defined indeed, and nowadays the concept of the "rigid body" has ceased to play any vital part in physical science.

Still, we do maintain that we can measure bodies, and, of course, our material standards can serve as measurers of time also, by adopting the velocity of light "in natural measure" as our unit of velocity. In laying down a co-ordinate system, we introduce no statements about measurement. Expressions for intervals of length, time, or separation can afterwards be introduced in terms of these co-ordinates, but the usefulness or validity of these can only be tested by the employment of material standards or "gauges," which are themselves part of the observed world. Of course, the procedure is an easy one apparently; we set up a standard gauge at one place, and then transport it, or accurately made copies of it, to all other places where required. But supposing that the journey produces some definite effect on it; not only so, but suppose that the effect is not the same for different ways of making the journey. Thus suppose two copies of the same gauge which fit accurately at A are transported to B, one by a path APB, the other by a different path AQB. What would be the result on the mathematical treatment of geometry if it were admitted that the gauges do not fit true at B? To be sure, our measures have not so far given any definite support to an assertion that they would not do so; but all measurement is in the last resort affected by some error, instrumental or personal, and we cannot certainly assert that they would fit to the most ultimate degree of accuracy. The point at present is not whether they do or do not fit, but what would be the result of admitting that possibly they do not. It must be grasped clearly that in speaking most generally of a different path, we consider time as well as space. Thus both gauges might actually be transported along the same path in the frame of reference, but one might pass through the intermediate positions at quite different times to the other; that would constitute difference of path from our enlarged point of view.

Such an admission would appear to leave our measuring system extremely arbitrary. We could, of course, set up a standard gauge at A and then define the routes along which the copies should be transported to other places where measurements are to be made. But naturally the question would

arise—on what grounds are we to select these privileged routes? And from that it is only a step to the query—why should there be any privileged routes at all? We have refused to endow any system of co-ordinates with special rights; why do so to any system of gauges? Is it not possible to express geometrical facts in a manner which renders these expressions as independent of a gauge system as they are of a co-ordinate system. Weyl has shown that it is possible, and has pointed out that the new geometrical quantities introduced for the purpose of such expression appear to be related to the potential functions of the electromagnetic field in a manner similar to that which connects the metric quantities $g_{\mu\nu}$ with the gravitational potentials of Einstein.

The connection between Riemann's geometry and the new one suggested by Weyl can be made clear in a simple way. In Riemann's geometry the transport of a vector round a closed path in a multi-dimensional continuum by successive elementary parallel displacements does not in general preserve the original direction of the vector. Weyl's suggestion is that neither does it preserve its gauged magnitude, for the admission that two gauges which fit at A may not fit at B, after one has travelled along APB and the other along AQB, amounts to a statement that a gauge which fits a standard gauge at A may not fit it after travelling round a closed path APBQA in which the return journey from B is not the outward journey merely reversed. In short, there is a non-integrability of magnitude as well as of direction.

But in order to make any headway at all we must postulate that such a thing as a "congruent displacement" exists, just as we postulated that such a thing as a "parallel displacement" exists. To be sure, the vector A^λ at the point x_λ is not in general parallel to the vector A^λ at $x_\lambda + \delta x_\lambda$; so now by this further enlargement of our ideas the squared distance, which is gauged as l^2 at x_λ, is not congruent to the distance, whose square is gauged as l^2 at $x_\lambda + \delta x_\lambda$.* But there is a one-one correspondence between distances at x_λ and distances at $x_\lambda + \delta x_\lambda$. The vector at $x_\lambda + \delta x_\lambda$, parallel to A^λ at x_λ, was defined by

$$\delta A^\lambda = -f_\beta{}^\lambda \delta x_\beta,$$

where the $f_\mu{}^\lambda$ were assumed to be linear functions of the vector components, viz.,

$$f_\mu{}^\lambda = f_{\alpha\mu}{}^\lambda A^\alpha.$$

* As the sequel will show, it is a little more convenient to speak in terms of the square of a distance rather than the distance itself.

So now the distance at $x_\lambda + \delta x_\lambda$ congruent to a distance whose square is gauged as l^2 at x_λ, is gauged as $l^2 + \delta l^2$ where δl^2 is defined by

$$\delta l^2 = l^2 \phi_a \delta x_a, \quad . \quad . \quad . \quad . \quad (1)$$

$\phi_1, \phi_2, \phi_3, \phi_4$ being four scalar functions of position. This assumes that the alteration in gauging is linearly related to the length itself, which seems a necessary condition if displacements are to preserve similarity between a set of distances at x_λ and the congruent distances at $x_\lambda + \delta x_\lambda$.

But, as Eddington has pointed out, there is still a wider assumption possible, and that in postulating that the multiplier of l^2, which gives the change in the measure, is proportional to l^2, Weyl has still left a restriction on the geometrical treatment of the space-time continuum which it may be necessary to remove. We shall return to this point later, however, and proceed to develop the consequences of (1).

As $\phi_a \delta x_a$ must be independent of a change of co-ordinates, it follows that ϕ_λ is a covariant vector.

It is clear that the formulæ of Chapter XIV. will require modification in view of Weyl's assumption, notably the expressions for the quantities $f_{\lambda\mu}{}^\nu$ and the curvature tensors. Thus it will be found that $f_{\lambda\mu}{}^\nu$ contains terms in addition to $\{\lambda\mu, \nu\}$, and that the Riemann-Christoffel tensor is only a part of the expression required in Weyl's geometry. We shall employ the symbols $f_{\lambda\mu}{}^\nu$ and $f_{\lambda\mu\nu}$ to denote these general values, but in accordance with the practice in recent literature, we shall indicate a tensor generalised from the corresponding tensor of the Riemann theory by prefixing an asterisk.

On referring to Chapter XIV., it will be observed that in the work leading to equation (12) of that chapter we can no longer put

$$(\partial g_{\alpha\beta}/\partial x_\epsilon - f_{\beta\epsilon\alpha} - f_{\alpha\epsilon\beta}) A^\alpha A^\beta \delta x_\epsilon$$

equal to zero. We must equalise it with the right-hand side of the equation (1) just written, where l^2, being the invariant squared magnitude of the vector A^λ, is given by

$$l^2 = g_{\alpha\beta} A^\alpha A^\beta \quad . \quad . \quad . \quad . \quad (2)$$

Hence

$$(\partial g_{\alpha\beta}/\partial x_\epsilon - f_{\beta\epsilon\alpha} - f_{\alpha\epsilon\beta}) A^\alpha A^\beta \delta x_\epsilon = g_{\alpha\beta} A^\alpha A^\beta \phi_\epsilon \delta x_\epsilon.$$

In consequence of this

$$\partial g_{\lambda\mu}/\partial x_\nu - f_{\mu\nu\lambda} - f_{\lambda\nu\mu} = g_{\lambda\mu}\phi_\nu, \quad . \quad . \quad (3)$$

whence it follows by steps similar to those in Chapter XIV. that

$$f_{\lambda\mu\nu} = [\lambda\mu,\ \nu] + \tfrac{1}{2}(g_{\lambda\nu}\phi_\mu + g_{\mu\nu}\phi_\lambda - g_{\lambda\mu}\phi_\nu) \qquad . \quad (4)$$

and

$$f_{\lambda\mu}{}^{\nu} = \{\lambda\mu,\ \nu\} + \tfrac{1}{2}(g_\lambda{}^\nu\phi_\mu + g_\mu{}^\nu\phi_\lambda - g_{\lambda\mu}\phi^\nu). \qquad . \quad (5)$$

On developing the curvature tensor by the method of Chapter XIV., we find that after a parallel displacement round a *small* circuit the change in the component A^κ, i.e. ΔA^κ, is still given by (15) of that chapter, where the tensor is still formally defined as in (16), but with the extended expression for $f_{\lambda\mu}{}^\nu$ written above.

Thus

$$\Delta A^\kappa = \tfrac{1}{2}{}^*R_{\alpha\beta\gamma}{}^\kappa A^\alpha \delta S^{\beta\gamma}, \qquad . \qquad . \qquad . \quad (6)$$

where

$$\left.\begin{aligned}{}^*R_{\lambda\mu\nu}{}^\kappa &= \partial f_{\lambda\nu}{}^\kappa/\partial x_\mu - \partial f_{\lambda\mu}{}^\nu/\partial x_\nu + f_{\lambda\nu}{}^\alpha f_{\alpha\mu}{}^\kappa - f_{\lambda\mu}{}^\alpha f_{\alpha\nu}{}^\kappa \\ &= R_{\lambda\mu\nu}{}^\kappa + \partial U_{\lambda\nu}{}^\kappa/\partial x_\mu - \partial U_{\lambda\mu}{}^\kappa/\partial x_\nu \\ &\quad + \{\lambda\nu\ \alpha\}U_{\alpha\mu}{}^\kappa + \{\alpha\mu,\ \kappa\}U_{\lambda\nu}{}^\alpha + U_{\lambda\nu}{}^\alpha U_{\alpha\mu}{}^\kappa \\ &\quad - \{\lambda\mu,\ \alpha\}U_{\alpha\nu}{}^\kappa - \{\alpha\nu,\ \kappa\}U_{\lambda\mu}{}^\alpha - U_{\lambda\mu}{}^\alpha U_{\alpha\nu}{}^\kappa,\end{aligned}\right\} \quad (7)$$

where we write $U_{\lambda\mu\nu}$ for the tensor (symmetric in λ and μ)

$$\tfrac{1}{2}(g_{\lambda\nu}\phi_\mu + g_{\mu\nu}\phi_\lambda - g_{\lambda\mu}\phi_\nu)$$

and

$$U_{\lambda\mu}{}^\nu = \tfrac{1}{2}(g_\lambda{}^\nu\phi_\mu + g_\mu{}^\nu\phi_\lambda - g_{\lambda\mu}\phi_\nu). \qquad . \qquad . \quad (8)$$

Employing the notation $(A_\lambda)_\mu$ still to represent covariant derivation, as defined in Chapter IX., i.e., as a device for deriving tensors by differentiation without any consideration of gauges,

$$\begin{aligned}{}^*R_{\lambda\mu\nu}{}^\kappa = R_{\lambda\mu\nu}{}^\kappa + (U_{\lambda\nu}{}^\kappa)_\mu - (U_{\lambda\mu}{}^\kappa)_\nu + U_{\lambda\nu}{}^\alpha U_{\alpha\mu}{}^\kappa \\ - U_{\lambda\mu}{}^\alpha U_{\alpha\nu}{}^\kappa. \quad (9)\end{aligned}$$

Putting $\kappa = \beta$, multiplying by $g_{\kappa\beta}$, and contracting we obtain

$$\begin{aligned}{}^*R_{\lambda\mu\nu\kappa} = R_{\lambda\mu\nu\kappa} + (U_{\lambda\nu\kappa})_\mu - (U_{\lambda\mu\kappa})_\nu + U_{\lambda\nu}{}^\alpha U_{\alpha\mu\kappa} \\ - U_{\lambda\mu}{}^\alpha U_{\alpha\nu\kappa}. \quad (10)\dagger\end{aligned}$$

The tensor $R_{\lambda\mu\nu\kappa}$ is anti-symmetric as regards the indices μ and ν, and inspection will show that the remaining terms on the right-hand side of (10) are also anti-symmetric with respect to μ and ν. Hence this is also true of the tensor ${}^*R_{\lambda\mu\nu\kappa}$. It is another matter, however, when we consider κ and λ. Certainly

† The reader must observe that the lowering of the index κ inside the bracket which indicates covariant differentiation, is not purely mechanical as in the other terms, but requires a few steps which, however, are not difficult and involve the equation, $g_{\kappa\beta}\{\lambda\mu,\ \beta\} = [\lambda\mu,\ \beta]$.

$R_{\lambda\mu\nu\kappa}$ is anti-symmetric with regard to those indices, but the remaining terms are not. However, they can be separated into two tensors, one symmetric with respect to κ and λ, and one anti-symmetric, for any tensor such as $A_{\kappa\lambda}$ can be written as

$$\tfrac{1}{2}(A_{\kappa\lambda} + A_{\lambda\kappa}) + \tfrac{1}{2}(A_{\kappa\lambda} - A_{\lambda\kappa}),$$

and the first bracket defines a symmetric tensor, the second an anti-symmetric. In this way it is clear that we can write

$$\begin{aligned} {}^*R_{\lambda\mu\nu\kappa} &= R_{\lambda\mu\nu\kappa} + H_{\lambda\mu\nu\kappa} + K_{\lambda\mu\nu\kappa} \\ &= I_{\lambda\mu\nu\kappa} + K_{\lambda\mu\nu\kappa}, \end{aligned}$$

where $H_{\lambda\mu\nu\kappa}$ and $K_{\lambda\mu\nu\kappa}$ are the two parts of the expression formed by the last four terms on the right-hand side of (10), anti-symmetric and symmetric in κ and λ respectively; and $I_{\lambda\mu\nu\kappa}$ is the tensor $R_{\lambda\mu\nu\kappa} + H_{\lambda\mu\nu\kappa}$, and is anti-symmetric with regard to κ and λ as well as μ and ν. The value of $K_{\lambda\mu\nu\kappa}$ in terms of the $g_{\lambda\mu}$ and ϕ_λ can be worked out by a series of tedious steps; but it can be arrived at very readily by the following artifice.

Consider the change in l^2 the squared magnitude of the vector A^λ after parallel displacement round a small path. On the one hand, we have by (1)

$$\left.\begin{aligned} \Delta l^2 &= l^2 \textstyle\int \phi_\alpha dx_\alpha \\ &= l^2 \Phi_{\alpha\beta}\delta S^{\alpha\beta} \end{aligned}\right\} \quad . \quad . \quad . \quad (12)$$

by Stokes' Theorem, where

$$\Phi_{\lambda\mu} = \partial\phi_\mu/\partial x_\lambda - \partial\phi_\lambda/\partial x_\mu,$$

and $\delta S^{\lambda\mu}$ is the anti-symmetric tensor corresponding to the element of area enclosed by the path.

On the other hand,

$$\begin{aligned} \Delta l^2 &= \Delta(g_{\alpha\beta}A^\alpha A^\beta) \\ &= \Delta(A_\alpha A^\alpha) \\ &= A_\alpha \Delta A^\alpha + A^\alpha \Delta A_\alpha \\ &= 2A_\alpha \Delta A^\alpha, \end{aligned}$$

since ΔA^λ is a vector (because it is the difference of two vectors at the same point), and in consequence $\Delta A_\lambda = g_{\lambda\alpha}\,\Delta A^\alpha$. Thus by (6)

$$\begin{aligned} \Delta l^2 &= A_\alpha {}^*R_{\beta\gamma\epsilon}{}^{\alpha} A^\beta \delta S^{\gamma\epsilon} \\ &= {}^*R_{\beta\gamma\epsilon\alpha} A^\alpha A^\beta \delta S^{\gamma\epsilon} \\ &= {}^*R_{\gamma\alpha\beta\epsilon} A^\gamma A^\epsilon \delta S^{\alpha\beta} \end{aligned}$$

by a change of dummy suffixes.

But this is equal to

$$K_{\gamma\alpha\beta\epsilon}A^{\gamma}A^{\epsilon}\delta S^{\alpha\beta},$$

for the anti-symmetry of the tensor $I_{\gamma\alpha\beta\epsilon}$ in γ and ϵ entails its disappearance in the summation.

Hence

$$\begin{aligned} K_{\gamma\alpha\beta\epsilon}A^{\gamma}A^{\epsilon} &= \Phi_{\alpha\beta}l^2 \\ &= \Phi_{\alpha\beta}g_{\gamma\epsilon}A^{\gamma}A^{\epsilon}, \end{aligned}$$

i.e.

$$K_{\lambda\mu\nu\kappa} = g_{\kappa\lambda}\Phi_{\mu\nu}. \quad . \quad . \quad . \quad (13)$$

Returning to equation (9), we put $\kappa = \nu = \beta$ and contract. The result is

$$*G_{\lambda\mu} = G_{\lambda\mu} + (U_{\lambda\beta}{}^{\beta})_{\mu} - (U_{\lambda\mu}{}^{\beta})_{\beta} + U_{\lambda\beta}{}^{\alpha}U_{\mu\alpha}{}^{\beta} - U_{\lambda\mu}{}^{\alpha}U_{\alpha\beta}{}^{\beta}. \quad (14)$$

The only term on the right-hand side which is not symmetric in λ and μ is the second term, and we can easily show that

$$U_{\lambda\beta}{}^{\beta} = 2\phi_{\lambda},$$

so that the non-symmetric term in (14) is $2(\phi_{\lambda})_{\mu}$.

Thus

$$*G_{\lambda\mu} = G_{\lambda\mu} + V_{\lambda\mu} + 2(\phi_{\lambda})_{\mu}$$

where $V_{\lambda\mu}$ is a symmetric tensor.

This can be written

$$\left.\begin{aligned} *G_{\lambda\mu} &= G_{\lambda\mu} + V_{\lambda\mu} + (\phi_{\lambda})_{\mu} + (\phi_{\mu})_{\lambda} + (\phi_{\lambda})_{\mu} - (\phi_{\mu})_{\lambda} \\ &= G_{\lambda\mu} + H_{\lambda\mu} + K_{\lambda\mu} \\ &= I_{\lambda\mu} + K_{\lambda\mu} \end{aligned}\right\} \quad (15)$$

where $I_{\lambda\mu}$ is a symmetric tensor and $K_{\lambda\mu}$ is the anti-symmetric tensor

$$\begin{aligned} &(\phi_{\lambda})_{\mu} - (\phi_{\mu})_{\lambda} \\ &= \partial\phi_{\lambda}/\partial x_{\mu} - \partial\phi_{\mu}/\partial x_{\lambda} \\ &= -\Phi_{\lambda\mu}. \end{aligned}$$

This result agrees with (13), for since

$$K_{\lambda\mu\nu\kappa} = g_{\kappa\lambda}\Phi_{\mu\nu},$$

we have

$$K_{\lambda\mu\nu}{}^{\kappa} = g^{\kappa}{}_{\lambda}\Phi_{\mu\nu}$$

and therefore

$$\begin{aligned} K_{\lambda\mu} &= g^{\alpha}{}_{\lambda}\Phi_{\mu\alpha} \\ &= \Phi_{\mu\lambda} \\ &= -\Phi_{\lambda\mu}. \end{aligned}$$

A short digression may be made here to point out that certain

tensors possess the property of being unchanged by any alteration in the gauge system. Thus suppose that the gauges are changed in such a manner that an interval which was formerly gauged as l at a world-point is now gauged as $l\psi^{\frac{1}{2}}$ *at the same point*, ψ being some function of position. In consequence of this the gauge-vector ϕ_λ will be altered to ϕ_λ' where

$$\delta(\psi l^2) = \psi l^2 \phi_a' \delta x_a.$$

So that

$$\begin{aligned} \phi_a' \delta x_a &= \delta(\psi l^2)/\psi l^2 \\ &= \delta l^2/l^2 + \delta\psi/\psi \\ &= \phi_a \delta x_a + \delta \log \psi. \end{aligned}$$

Hence

$$\phi_\lambda' = \phi_\lambda + \partial \log \psi/\partial x_\lambda. \quad . \quad . \quad (16)$$

In consequence

$$\begin{aligned} \Phi_{\lambda\mu}' &= \partial\phi_\mu'/\partial x_\lambda - \partial\phi_\lambda'/\partial x_\mu \\ &= \partial\phi_\mu/\partial x_\lambda - \partial\phi_\lambda/\partial x_\mu \\ &= \Phi_{\lambda\mu}. \end{aligned}$$

$\Phi_{\lambda\mu}$ is thus such a tensor as we have just referred to. It may be called an " absolute " tensor, but the phrase " in-tensor " has been introduced by Eddington to distinguish it from tensors which do not possess this property.

A little thought will show that $*R_{\lambda\mu\nu}{}^\kappa$ is also an in-tensor. In the equation (6), A^λ is an " in-vector " ; for it is a displacement, i.e., a difference of co-ordinates, and in no way depends on a gauge. Likewise, $\delta S^{\lambda\mu}$ depends only on co-ordinate differences. So an alteration of gauge will not affect $*R_{\lambda\mu\nu}{}^\kappa$. But this is not so for $*R_{\lambda\mu\nu\kappa}$. To see this, we notice first of all that $g_{\mu\nu}$ is not an in-tensor, for if the change of gauge alters a measured interval from l^2 to ψl^2, i.e., from $g_{\alpha\beta}\delta x_\alpha \delta x_\beta$ to $\psi g_{\alpha\beta}\delta x_\alpha\delta x_\beta$, the altered $g_{\mu\nu}'$ will be $\psi g_{\mu\nu}$. Now

$$*R'_{\lambda\mu\nu}{}^\kappa = *R_{\lambda\mu\nu}{}^\kappa$$

(remembering that the accent refers to a change of gauges and not of co-ordinates).

Hence

$$\begin{aligned} *R'_{\lambda\mu\nu\kappa} &= g'_{\kappa\alpha} *R'_{\lambda\mu\nu}{}^\alpha \\ &= \psi g_{\kappa\alpha} *R_{\lambda\mu\nu}{}^\alpha \\ &= \psi *R_{\lambda\mu\nu\kappa}. \end{aligned}$$

It follows that $g_{\mu\nu}$ and $*R_{\lambda\mu\nu\kappa}$ are not in-tensors ; Weyl calls them tensors of " weight " unity. In general, if the tensor after the change of gauge-system has a value equal to product of its former value by ψ^n, its weight is n. Thus in-tensors

have the weight zero. As the determinant g is altered to $\psi^4 g$ by the change in gauge, and each first minor is also multiplied by ψ^3, it appears that $g^{\mu\nu}$ has the weight minus unity. In consequence, lowering an index of a mixed or contravariant tensor by association produces a tensor whose weight is larger by unity; while raising an index of a mixed or covariant tensor reduces the weight by unity. For example, $\Phi^{\lambda\mu}$ is a contravariant anti-symmetric tensor of weight minus two. Hence $\Phi_{\alpha\beta}\Phi^{\alpha\beta}$ is an invariant (for change of co-ordinates) of weight minus two. On the other hand,

$$\int q\Phi_{\alpha\beta}\Phi^{\alpha\beta}d\omega$$

is an "in-invariant," since $d\omega$ depends only on co-ordinate differences and the quantity q or $\sqrt{(-g)}$ has a weight two.

We are now in a position to understand the suggestion which has been made by Weyl. He points out that the abandonment of the geometrical postulate that a parallel displacement round a closed path preserves a direction unchanged involves the introduction of the metric coefficients $g_{\mu\nu}$, and gives us gravitation by the identification of the potentials of the gravitational field with these coefficients. He proposes the theory that the abandonment of the geometrical postulate that congruent displacement round a closed path preserves a length unchanged will give us electromagnetism by a similar identification of the potentials of the electromagnetic field with the four gauge-coefficients. The suggestion is certainly a fascinating one. We have seen how fourteen quantities are required by the physicist to define the state of the world at any point-instant, viz., the ten gravitational potentials and the four electromagnetic potentials. Ten of these have been identified with ten metric coefficients, and experiment has supported the identification; the removal of an entirely unnecessary restriction on our geometry provides us with four more metric coefficients whose existence had not been thought of, just the number of physical quantities hitherto outside a complete identification. It is not unnatural to regard this as more than an accident and proceed to the identification suggested by Weyl. Unfortunately, no direct test of the theory appears to be feasible. It is entirely out of the question to attempt to measure the discrepancy in the length of a body after it has been carried round any ordinary circuit even a considerable number of times in any obtainable electromagnetic field. For, of course, the identification of the electromagnetic potentials with the ϕ_λ-coefficients identifies the field tensor with $\Phi_{\lambda\mu}$, and thus the discrepancy predicted

would vanish where there is no electromagnetic field, and increase in amount the stronger the field in which the circuit is situated. (See equation 12.) Weyl's hypothesis, however, is attractive to the relativist for a cogent reason—it does away with the privileged position which gravitation has hitherto occupied in the general theory of Relativity. It is quite conceivable that the laws of gravitation might have been expressed in tensor equations quite different from those given by Einstein; such laws would still have been compatible with General Relativity. It is even conceivable that the equations might have actually had the same form as Einstein's, except that the ten gravitational potentials might not have been the components of the fundamental tensor which determines the element of separation in space-time. Relativity would have had nothing to say for or against such equations; it would have been a matter for the experimentalist and astronomer. In such a case gravitational theory and electromagnetic theory would have been on an equal footing; they would both have passed the Relativity test, but to all appearances there would have been no obvious connection between the physical tensors involved in the formulation of these theories and any geometrical tensors involved in the development of the geometry of the world. But, as stated, the close connection between the gravitational potentials and the ten metric coefficients $g_{\mu\nu}$, postulated by Einstein, gave a position of privilege to gravitation not accorded to other physical forces. As the latter forces are nowadays reduced to terms of electromagnetic action, it is all to the good, as far as the opinion of those who desire elegance and comprehensiveness in physical theory has weight, if we can without any contradiction of observed fact place electromagnetism on the same plane as gravitation; and this Weyl has done by identifying the electromagnetic potentials with the ϕ_λ coefficients.

Eddington, however, has pointed out that even Weyl's geometry retains a restrictive feature whose removal abolishes the necessity of connecting electromagnetism with a theoretical change in the length of a transported measuring rod. Going back to the work which leads to equation (3) above, it is clear that in proving that

$$(\partial g_{\alpha\beta}/\partial x_\epsilon - f_{\beta\epsilon\alpha} - f_{\alpha\epsilon\beta})A^\alpha A^\beta \delta x_\epsilon$$

is an invariant (for it the difference of two invariants at the same world-point), we have proved that

$$\partial g_{\alpha\beta}/\partial x_\epsilon - f_{\beta\epsilon\alpha} - f_{\alpha\epsilon\beta}$$

is the component of a covariant tensor of the third order, since $A^{\alpha}A^{\beta}\delta x_{\epsilon}$ is a component of a contravariant tensor of the third order. But, of course, it does not prove that this covariant tensor is of any special type, such as the outer product of three covariant vectors, or the outer product of a covariant vector and a covariant tensor of the second order, as Weyl assumes since he equates the component to $g_{\alpha\beta}\phi_{\epsilon}$. Eddington simply starts from the fact that it is a covariant tensor of the third order, and puts no limitations on its form; he abandons the definition contained in (1), and proceeds as follows. Since

$$\begin{aligned} &\partial g_{\lambda\mu}/\partial x_{\nu} - f_{\mu\nu\lambda} - f_{\lambda\nu\mu} \\ &\partial g_{\lambda\nu}/\partial x_{\mu} - f_{\nu\mu\lambda} - f_{\lambda\mu\nu} \\ &\partial g_{\mu\nu}/\partial x_{\lambda} - f_{\nu\lambda\mu} - f_{\mu\lambda\nu} \end{aligned}$$

are each tensors of the third order, the sum of the second two minus the first is also a tensor. Hence we find that

$$f_{\lambda\mu\nu} - [\lambda\mu, \nu]$$

is a covariant tensor of the third order *symmetric in λ and μ.* Denote it by $U_{\lambda\mu\nu}$, where, of course, it is no longer limited in form as it was above, when it was expressed in terms of $g_{\lambda\mu}$ and ϕ_{λ}; indeed, no quantities corresponding to ϕ_{λ} have been so far introduced.

We have therefore

$$f_{\lambda\mu\nu} = [\lambda\mu, \nu] + U_{\lambda\mu\nu}$$

and

$$f_{\lambda\mu}{}^{\nu} = \{\lambda\mu, \nu\} + U_{\lambda\mu}{}^{\nu}.$$

From this we proceed to the introduction and definition of the in-tensor $*R_{\lambda\mu\nu}{}^{\kappa}$, and the tensor of weight unity $*R_{\lambda\mu\nu\kappa}$, as in (9) and (10), except that $U_{\lambda\mu\nu}$ is not now restricted to be a special tensor of the type defined in (8), an equation which now drops out of the analysis.

As before, $*R_{\lambda\mu\nu\kappa}$ is anti-symmetric with respect to μ and ν, and can be split into two parts—one $I_{\lambda\mu\nu\kappa}$, anti-symmetric with regard to κ and λ, and the other $K_{\lambda\mu\nu\kappa}$, symmetric with respect to κ and λ, and it is not difficult to prove that

$$K_{\lambda\mu\nu\kappa} = \tfrac{1}{2}[(f_{\nu\lambda\kappa} + f_{\nu\kappa\lambda})_{\mu} - (f_{\mu\lambda\kappa} + f_{\mu\kappa\lambda})_{\nu}]. \quad . \quad (17)$$

When we come to the contraction of $*R_{\lambda\mu\nu}{}^{\kappa}$, we obtain (14) as before. Of course,

$$U_{\lambda\alpha}{}^{\alpha}$$

is a covariant vector, and we denote it by the symbol

$$-2B_{\lambda}.$$

So that (14) can now be written

$$^*G_{\lambda\mu} = G_{\lambda\mu} - 2(B_\lambda)_\mu - (U_{\lambda\mu}{}^\alpha)_\alpha + U_{\lambda\alpha}{}^\beta U_{\mu\beta}{}^\alpha + 2U_{\lambda\mu}{}^\alpha B_\alpha. \quad (18)$$

For the further contraction to $^*G$, it will be found necessary to introduce a symbol to represent the covariant vector $U^\alpha{}_{\alpha\lambda}$, which is quite distinct from the vector $U_{\lambda\alpha}{}^\alpha$, for

$$U_{\lambda\mu}{}^\nu = g^{\nu\alpha} U_{\lambda\mu\alpha}$$

while

$$U^\lambda{}_{\mu\nu} = g^{\lambda\alpha} U_{\alpha\mu\nu}.$$

Let us denote $U^\alpha{}_{\alpha\lambda}$ by C_λ; and, in consequence, $U^\alpha{}_\alpha{}^\lambda$ by C^λ. Proceeding from (18) we obtain

$$\left.\begin{aligned} ^*G^\lambda{}_\mu &= G^\lambda{}_\mu - 2g^{\lambda\alpha}(B_\alpha)_\mu - g^{\lambda\beta}(U_{\beta\mu}{}^\alpha)_\alpha + U^\lambda{}_\alpha{}^\beta U_{\mu\beta}{}^\alpha \\ &\qquad\qquad + 2U^\lambda{}_\mu{}^\alpha B_\alpha \\ &= G^\lambda{}_\mu - 2(B^\lambda)_\mu - (U^\lambda{}_\mu{}^\alpha)_\alpha + U^\lambda{}_\alpha{}^\beta U_{\mu\beta}{}^\alpha + 2U^\lambda{}_\mu{}^\alpha B_\alpha \end{aligned}\right\} \quad (19)$$

(In obtaining this line we have to remember that the covariant derivative of $g^{\lambda\mu}$ is zero.)

Hence, putting $\lambda = \mu$ and contracting,

$$^*G = G - 2(B^\alpha)_\alpha - (C^\alpha)_\alpha + 2B_\alpha C^\alpha + U^{\gamma\alpha\beta} U_{\gamma\beta\alpha}. \quad (20)$$

As before, we can write

$$\left.\begin{aligned} ^*G_{\lambda\mu} &= G_{\lambda\mu} + H_{\lambda\mu} + K_{\lambda\mu} \\ &= I_{\lambda\mu} + K_{\lambda\mu} \end{aligned}\right\} \quad . \quad . \quad (21)$$

where $H_{\lambda\mu}$ is symmetric in λ and μ, and, therefore, $I_{\lambda\mu}$ as well, while $K_{\lambda u}$ is anti-symmetric.

As $-2(B_\lambda)_\mu$ is the only non-symmetric term on the right-hand side of (18), we get

$$H_{\lambda\mu} = -(B_\lambda)_\mu - (B_\mu)_\lambda - (U_{\lambda\mu}{}^\alpha)_\alpha + U_{\lambda\alpha}{}^\beta U_{\mu\beta}{}^\alpha + 2U_{\lambda\mu}{}^\alpha B_\alpha \quad (22)$$

and

$$\left.\begin{aligned} K_{\lambda\mu} &= (B_\mu)_\lambda - (B_\lambda)_\mu \\ &= \partial B_\mu/\partial x_\lambda - \partial B_\lambda/\partial x_\mu \end{aligned}\right\} \quad . \quad . \quad (23)$$

Since $B_\lambda = -\frac{1}{2} U_{\lambda\alpha}{}^\alpha$, it is easy to prove that

$$K_{\lambda\mu} = \tfrac{1}{2}(\partial f_{\lambda\alpha}{}^\alpha/\partial x_\mu - \partial f_{\mu\alpha}{}^\alpha/\partial x_\lambda). \quad . \quad . \quad (23')$$

(In Weyl's theory

$$U_{\lambda\mu\nu} = \tfrac{1}{2}(g_{\lambda\nu}\phi_\mu + g_{\mu\nu}\phi_\lambda - g_{\lambda\mu}\phi_\nu)$$

and it is easy to prove that with this restriction

$$B_\lambda = -\phi_\lambda$$

and

$$C_\lambda = -\phi_\lambda.)$$

The most arresting feature, however, of Eddington's investigation is not the analysis just outlined, but the manner in which he proposes to use it for the representation of physical facts. The older physics laid great stress on the development of mechanisms which would explain all processes of nature in terms of the behaviour of matter in bulk. That feature, while not exactly disappearing, is certainly one of diminishing importance, except for pedagogic purposes. Einstein's theory, for example, has nothing to say concerning mechanisms explaining gravitation. What it does is to "explain" gravitation in terms of two principles, the principle of Relativity and the principle of Equivalence, through a powerful mathematical calculus. Nothing more is actually needed to arrive at the necessary differential equations. Yet there is something of the arbitrary in Einstein's choice of the law of gravitation among other possible and apparently reasonable tensor equalities which might be suggested. It is justified because it is probably the simplest possible, and it "works." Now we notice that, whether we know this calculus or not, resort to geometrical illustration is very helpful, and we can find geometrical statements which run parallel with the development of the tensor calculus itself. At once arises the question, is there a closer connection between these geometrical illustrations and the processes of nature than was thought of when we entered on our task of extending the principle of Relativity to cover all the great generalisations of Physics? We began with the well-known and defined concepts of physical science, and brought them one by one within the grip of our tensor calculus, introducing tensors such as the matter and electromagnetic tensors, wherever required, defined in terms of these concepts. Einstein and his followers discovered equalities between these tensors and other tensors such as the $G_{\mu\nu}$, which are purely geometrical and involve only the quantities required to define an interval. These equalities "work." But what shall we say to the suggestion that when we survey the completed scheme we shall be compelled to admit that for the most comprehensive grasp of the "nature of things" we must reverse the order of procedure? We shall begin, it is suggested, with a four-dimensional continuum, and lay down a general quadratic expression for the squared distance between two contiguous points. Thereafter, the mathematician can develop a whole series of propositions by means of the tensor calculus. Certain *identities* make their appearance, some of which bear a remarkable resemblance to the form of the equations which are satisfied

by the quantities introduced by the physicist in his discussion of material phenomena (i.e., in his study of the four-dimensional space-time of physical events), equations which the pure mathematician did not consider in the development of his geometry. Is this an accident, or are the relations of equality set up between the material tensors of the physicist and the geometrical tensors of the mathematician nothing more than identities, and have the physicist and mathematician been approaching reality from different aspects, and been talking about the same thing in different terms ?

This is " Geometrisation of Physics " with a vengeance. It almost seems as if each individual will accept this position or not according to temperament. The reasons advanced for its acceptance seem to carry no weight in certain quarters.* Yet this book would be incomplete without some reference to this aspect of our subject. If the reader wishes to see it set forth with great vigour and clarity, he cannot do better than read Chapter XII. on " The Nature of Things " in Eddington's book, " Space, Time, and Gravitation," and also his papers referred to at the beginning of this chapter. For what immediately follows the writer is entirely indebted to these publications whose author possesses a gift of picturesque writing as powerful as his insight into the meaning of geometrical processes.

We have in a previous part of this chapter completed the development of a pure geometry of a very general kind. This is one of the points of vantage from which we can start in our attack on the problem of Nature. The other starting-point is our inductive study of observational science.

We then proceed to the setting up of a physical theory which is based upon *identification* of the geometrical functions with quantities obtained by experimental measurement.

Where can we find the first point of contact between our two avenues of approach ? Following Weyl, we have to admit that a gauge-system is as arbitrary as a co-ordinate system, so far as the geometry is concerned ; but on the observational side we have good reason to believe that there is a *natural* gauge-system, otherwise our attempts to identify lengths on distant objects like the sun and planets with lengths on the earth would lead to discordant and paradoxical results. Is it, perchance, in the successful identification of this natural gauge-system that we can begin the series of identifications

* See, for example, a paper by More in the " Phil. Mag.," Nov. 1921.

which, according to the view expressed above, constitutes physical theory? Actual physical events have probably no regular relation connecting them at all, none having any element of simplicity at all events, but on averaging over a group of contiguous events it may be possible that we can verify the existence, for the average components of displacement from one event to another, of a quadratic function, invariant for change of co-ordinates and change of gauges. This will necessitate the discovery of a covariant in-tensor of the second order by a method *which does not involve the introduction of a metric to begin with.* Now we do employ such a method in setting up the idea of parallelism or correspondence between displacements at different world-points, i.e. by introducing the coefficients $f_{\lambda\mu}{}^{\nu}$. By an obvious adaptation of the previous analysis we prove that

$$\tfrac{1}{2}{}^*R_{\alpha\beta\gamma}{}^{\kappa}A^{\alpha}\delta S^{\beta\gamma}$$

is the vector difference between the displacement A^{λ} before and after a parallel transference round a small closed circuit $\delta S^{\beta\gamma}$, where

$$^*R_{\lambda\mu\nu}{}^{\kappa} = \partial f_{\lambda\nu}{}^{\kappa}/\partial x_{\mu} - \partial f_{\lambda\mu}{}^{\kappa}/\partial x_{\nu} + f_{\lambda\nu}{}^{\alpha}f_{\alpha\mu}{}^{\kappa} - f_{\lambda\mu}{}^{\alpha}f_{\alpha\nu}{}^{\kappa}. \qquad (24)$$

Thus we establish the existence of the mixed in-tensor $^*R_{\lambda\mu\nu}{}^{\kappa}$. A contraction establishes the existence of an in-tensor of the second order $^*G_{\lambda\mu}$, and this can be resolved into a symmetric tensor $I_{\lambda\mu}$ and an anti-symmetric tensor $K_{\lambda\mu}$. Obviously

$$^*G_{\alpha\beta}A^{\alpha}A^{\beta} \quad . \quad . \quad . \quad . \quad (25)$$

is an in-invariant. On account of the anti-symmetry of $K_{\lambda\mu}$ it drops out in the summation, and

$$I_{\alpha\beta}A^{\alpha}A^{\beta} \quad . \quad . \quad . \quad . \quad (26)$$

is also an in-invariant expression related to two contiguous events. In the four-dimensional space of the mathematician the element of distance between two contiguous points is given by

$$g_{\alpha\beta}\delta x_{\alpha}\delta x_{\beta} \quad . \quad . \quad . \quad . \quad (27)$$

where $g_{\mu\nu}$ is a symmetric tensor.

Let us now identify the mathematician's four-dimensional continuum with the physicist's space-time. This means that the $I_{\mu\nu}$ of (26) must be identical with the $g_{\mu\nu}$ of (27), i.e.,

$$I_{\mu\nu} = g_{\mu\nu}. \quad . \quad . \quad . \quad . \quad (28)$$

Obviously (28) could not possibly remain true for any arbitrary change of gauges if true for an assigned gauge-system, which means that (28) defines a natural gauge-system. As Eddington says: " The natural gauge of the (physical) world is determined by measures of space and time made with material and optical appliances. Any apparatus used to measure the world is itself part of the world, so that a natural gauge represents the world as self-gauging."

The introduction of a metric leads, of course, to the development of the Christoffel indices, the ordinary curvature tensor, and the tensors $G_{\mu\nu}$, $U_{\lambda\mu\nu}$, and $H_{\mu\nu}$, and the proof that $K_{\mu\nu}$ is the curl of a vector.

Of course, (28) not only fixes a gauge-system, but also a unit of length. If we wish still to make use of the centimetre as a convenient unit, we replace (28) by

$$I_{\mu\nu} = \lambda g_{\mu\nu}, \quad . \quad . \quad . \quad . \quad (29)$$

where λ is some universal constant.

As we proceed in our series of identifications we shall see that $I_{\mu\nu}$ differs from $G_{\mu\nu}$ by a tensor which will be identified with a tensor representing electromagnetic and electronic effects. So that the " emptier " the space the closer will $I_{\mu\nu}$ approximate to $G_{\mu\nu}$. Hence in the " void "

$$G_{\mu\nu} = \lambda g_{\mu\nu}. \quad . \quad . \quad . \quad . \quad (30)$$

So Einstein's law of gravitation proves to be a "gauging equation."

The *ten* equations (30) agree with the view that the world is spherical. We know that *twenty* conditions must be satisfied by a truly spherical world, viz.,

$$R_{\lambda\mu\nu\kappa}/|\,\kappa\lambda,\ \mu\nu\,| = \text{constant}.$$

(See equation (16), Chapter XIV.) Nevertheless, a world conditioned by

$$G_{\mu\nu} = \lambda g_{\mu\nu}$$

will possess the more important spherical properties. In fact, it is we who give it this symmetrical form by our choice of the natural gauge. We are aware of the difficulty involved in transporting gauges. When we say that the lengths of two material objects at different places are equal, we mean that the length of one bears to a standard length at the same place and in the same direction, the same ratio as the length of the other bears to a standard length at the latter place and in its direction. Our gauging equation simply implies that we have chosen the

radius of curvature, of the finite world at each place and in each direction as the standard for any linear measure *at that place and in that direction,* and we have assigned for convenience a unit which makes this standard equal to $\sqrt{(3/\lambda)}$ cm. There is no such thing as an absolute size. " Any material object of specified constitution must have determined size and shape *in relation to the radii of the world.* And in so far as the measure of length rests on such bodies, the form of the world expressed in this measure is bound to be quasi-spherical." (Eddington.) For an amplification of the argument, see in particular the paper in the " Phil. Mag.," November, 1921.

As relativists we must, of course, see that the mathematical expression of this choice is a covariant equation. We have written down

$$G_{\mu\nu} = \lambda g_{\mu\nu}$$

as the mathematical expression, and it certainly is covariant. That it agrees with the condition that the curvature at all points, and in all directions, is uniform can be inferred from the arguments of Chapter XV. It can be shown directly by reference to the concluding paragraphs of Chapter XIV. Thus representing a four-dimensional manifold with general Riemannian curvature as a hypersurface in Euclidean space of higher dimensions, and using the five dimensions given by the tangent plane hypersurface and the normal, we have for the equation of the four-dimensional continuum near the origin

$$2z = k_1x_1^2 + k_2x_2^2 + k_3x_3^2 + k_4x_4^2.$$

By the methods indicated in Chapter XIV. we obtain, at the origin (which is entirely arbitrary),

$$G_{11} = - k_1(k_2 + k_3 + k_4),$$

similar results for G_{22}, G_{33}, G_{44}, and zero values for $G_{\mu\nu}$ where $\mu \neq \nu$. The condition $G_{\mu\nu} = \lambda g_{\mu\nu}$ at the origin becomes

$$G_{11} = G_{22} = G_{33} = G_{44} = \lambda,$$

because at the origin $g_{11} = g_{22} = g_{33} = g_{44} = 1$, and $g_{12} = 0 =$ etc.

Hence

$$k_1 = k_2 = k_3 = k_4 = \sqrt{(\lambda/3)},$$

which proves the statement.*

* In a paper contributed to the " Phil. Mag." Jan., 1922, Eddington points out that this proof is of too limited a character, and supplies one which takes account of the *ten* dimensions which must be possessed by

Of course, equation (30) is only true in empty space. There is intense local curvature in the regions occupied by electrons, and equation (29) is then true, there being under these circumstances large differences between $I_{\mu\nu}$ and $G_{\mu\nu}$.

We now proceed further with our scheme of identification. We have identified physical space-time with a Riemannian four-dimensional continuum, and we need trouble no further about gauges, as we have assigned a gauge-system; so we can be quite content if our tensors possess the ordinary properties of covariance for a change of co-ordinates alone. There is no need to restrict ourselves to in-tensors. Let us now identify "things." What geometrical tensors possess the properties of physical quantities *by virtue of mathematical identities*. Reviewing the content of physical science, we note three sorts of attributes to things which require identification:

(1) Energy, momentum, stress.
(2) Electromagnetic force-tensor.
(3) Electric charge and current.

(1) The energy tensor T_μ^ν must satisfy the equation of conservation

$$(T_\mu^\alpha)_\alpha = 0.$$

Among our geometrical tensors we find one which satisfies a similar equation, viz.,

$$G_\mu^\nu - \tfrac{1}{2}g_\mu^\nu(G - a),$$

where a is any constant, for we know that

$$(G_\mu^\alpha)_\alpha = \tfrac{1}{2}(g_\mu^\alpha(G - a))_\alpha$$

is an identity.

So we write in the first instance

$$T_\mu^\nu = k[G_\mu^\nu - \tfrac{1}{2}g_\mu^\nu(G - a)],$$

where k is a constant depending on units. The constant a is, of course, not arbitrary; it is readily determined, for in empty space T_μ^ν is to vanish, and so in empty space we have by contraction

$$-G + 2a = 0.$$

Hence in empty space

$$G_\mu^\nu = \tfrac{1}{2}ag_\mu^\nu.$$

the higher Euclidean space in order to give full scope to the "vagaries of which a four-dimensional surface is capable when it has six extra dimensions to twist about in."

Thus

$$a = 2\lambda,$$

and so

$$T_\mu^\nu = k[G_\mu^\nu - \tfrac{1}{2}g_\mu^\nu(G - 2\lambda)]. \qquad (31)$$

(2) The electromagnetic force-tensor $F_{\mu\nu}$ must satisfy

$$\partial F_{\mu\nu}/\partial x_\lambda + \partial F_{\nu\lambda}/\partial x_\mu + \partial F_{\lambda\mu}/\partial x_\nu = 0,$$

which is the tensor expressions of two of Maxwell's equations. The Curl of any covariant vector would serve our purpose here as far as the mathematical identity is concerned; but the appearance of the fundamental in-tensor $K_{\mu\nu}$ in the analysis of our curvature in-tensor into symmetric and anti-symmetric parts leads us naturally to the identification of $F_{\mu\nu}$ with $K_{\mu\nu}$, or of the electromagnetic potential vector with B_μ (or ϕ_μ in Weyl's more restricted geometry). We might, of course, identify the electromagnetic potential with C_μ and $F_{\mu\nu}$ with the Curl of C_μ; but this seems unnatural, in view of the actual appearance of $K_{\mu\nu}$ or Curl B_μ in the analysis of curvature. So we write

$$F_{\mu\nu} = fK_{\mu\nu}, \qquad (32)$$

where f is a constant depending on units.

(3) The current-charge vector J^μ must satisfy the conditions of conservation of charge, i.e.,

$$(J^\alpha)_\alpha = 0.$$

Now if $A^{\mu\nu}$ is *any* anti-symmetric covariant tensor, then

$$((A^{\alpha\beta})_\beta)_\alpha = 0$$

is an identity. Thus we might identify J^μ with $(A^{\mu\beta})_\beta$. We naturally seek for some important anti-symmetric tensor for $A^{\mu\nu}$, and $K^{\mu\nu}$ rises once more in our minds. This suggestion receives confirmation in the fact that if we identify the physical vector J^μ with the geometrical vector $(K^{\mu\alpha})_\alpha$, we obtain from (2) at once the *equations*

$$J^\mu = (F^{\mu\alpha})_\alpha,$$

which are the remaining pair of Maxwell's equations.

We can now consider the grounds on which Eddington regards Weyl's geometry as too limited.

In physical space-time the world-lines of all light pulses emitted from a point in space at a given instant, and in all

directions, form a conical locus whose equation near the corresponding world-point is

$$\delta s = 0$$

or
$$g_{\alpha\beta}\delta x_\alpha \delta x_\beta = 0.$$

Now this should be a unique locus in space-time at this point, and Weyl argues that this requires that a vector of zero magnitude should remain a vector of zero magnitude after transference round any closed circuit back to the same point. This condition is certainly satisfied in his theory, for by (12)

$$\Delta l^2 = l^2 \Phi_{\alpha\beta} \delta S^{\alpha\beta}$$

so that if l^2 is zero, Δl^2 is also zero.

This condition is not of necessity satisfied by Eddington's theory, because by a glance at the analysis immediately succeeding equation (12),

$$\Delta l^2 = K_{\gamma\alpha\beta\epsilon} A^\gamma A^\epsilon \delta S^{\alpha\beta},$$

and if we do not impose Weyl's form (13) on $K_{\lambda\mu\nu\kappa}$, Δl^2 need not be zero, even if A^λ is a vector of zero magnitude.† Yet Eddington argues that the condition that the light pulse locus should be unique can be satisfied in a more general way than that imposed by Weyl. If the locus exists independent of co-ordinates and gauges, we should seek for its analogue in four-dimensional geometry in some in-invariant equation defining a unique conical locus at a definite point in the four-dimensional continuum. This equation can only be

$$P_{\alpha\beta}\delta x_\alpha \delta x_\beta = 0,$$

where $P_{\mu\nu}$ is some in-tensor. There is only one possibility; $P_{\mu\nu}$ is $*G_{\mu\nu}$. Hence the geometrical locus is

$$*G_{\alpha\beta}\delta x_\alpha \delta x_\beta = 0,$$

or, since the anti-symmetric terms drop out, the locus is

$$I_{\alpha\beta}\delta x_\alpha \delta x_\beta = 0.$$

But since in our physical space-time the gauge has been chosen to satisfy

$$I_{\mu\nu} = \lambda g_{\mu\nu},$$

the process of identification gives us

$$g_{\alpha\beta}\delta x_\alpha \delta x_\beta = 0$$

† The reader should, of course, require no reminder that a zero magnitude is quite consistent with finite values for the components.

as the equation of the light pulse locus, and the necessities of physical science are satisfied.

Reflecting on the contents of the immediately preceding pages, we cannot fail to see that Eddington's wider geometry is consistent with a simpler physical theory than Weyl's. If we adopt the view that physical theory must be expressed by means of equations which are to remain invariant for both gauge and co-ordinate transformation, then these equations must employ *in-tensors* and not merely tensors, such as $G_{\mu\nu}$, which are altered by a change of gauges. ($G_{\mu\nu}$ would, of course, be unaltered for a limited group of gauge transformations—those in which the ψ function of equation (16) is a constant—but not for transformations of a perfectly arbitrary nature.) This means that Einstein's theory would be approximate and require generalisation, except in regions where there exists no electromagnetic field. Now there is nothing to be gained by complicating our physical theory if it can be avoided, and Eddington has shown that it is unnecessary. The parallel displacement of a vector is a process which may yield a change in the magnitude of the vector or not, just according to the postulates on which we decide to base our geometrical reasoning. In any event, we arrive at a tensor $*G_{\mu\nu}$ which provides us with a *unique* theoretical gauge. Without necessarily abandoning Weyl's *geometrical* postulate, we, nevertheless, contrive to avoid the awkward results in our *physical* theory which, at the first glance, it would seem to entail; for we can assume as a physical fact (until experiment contradicts it) that any material standard, in being transferred to a new position or rotated into a new direction, adjusts itself so as to be a *constant* fraction of the radius of curvature of the world at that place and in that direction. This assumption replaces in our physical theory the former assumption that in some manner the presence of an electromagnetic field involves a change in the size of a material body if it be moved about, even if it be returned to the place whence it started. It is clear that in view of the possibilities raised by our enlarged ideas concerning geometry, some physical postulate is required; and one which permits a material standard to retain an unambiguous size at a defined place and in a defined direction would, even before investigation of its consequences, appear to possess a decided advantage. It allows us to treat the magnitude of a body as something determined by its present situation and not modified by the persistence in it of the effects of its previous history. This postulate allows us also to retain Einstein's gravitational equations as *exact* state-

ments, for the physical theory has no concern with ambiguities of gauging; a gauge has been adopted for each place and direction, and Einstein's hypothesis of a unique and invariant separation between two assigned events is preserved. We have no need to enlarge our Relativity standpoint beyond that required by Einstein's theory. That theory was made possible by the abrogation of any privileged system of co-ordinates. On the other hand, we find it quite unnecessary to modify physical theory still further in order to suit the abandonment of a privileged gauge; on the contrary, we retain such a gauge in order to retain Einstein's theory. The purpose which Weyl had in view in adopting his original physical assumption was the derivation from the metric of a tensor whose nature rendered it suitable for identification with the electromagnetic field tensor, and the consequent destruction of the privilege hitherto accorded to gravitation in the Relativity theory. But that purpose is still achieved by Eddington's assumption of the natural gauge. The in-tensor $*G_{\mu\nu}$ yields the metric or gravitational tensor $I_{\mu\nu}$ or $g_{\mu\nu}$, and also $K_{\mu\nu}$, which has the properties just required for a tensor which satisfies Maxwell's equations. More than that, it gives yet another tensor $H_{\mu\nu}$, which, although not identified so far with any physical tensor, might prove useful in enlarging the scope of our physical theory if some one should by some ingenious suggestion use the method of identification not merely to recognise old friends, but to discover new acquaintances.

Anyone who has not a wide acquaintance with Riemannian geometry might reasonably ask why the physicist confines the identifying process to such a small number of geometric tensors. No doubt the consideration of curvature in its Riemannian sense has given rise to tensors satisfying mathematical identities required for the laws of gravitation and mechanics; and the generalisation of the treatment of curvature by Weyl and Eddington has provided us with the tensors required for the expression of the laws of electromagnetism. But after all, might there not exist other tensors having the required mathematical properties, and, if so, why should they not be employed in the geometrisation of physics? The answer is that the available tensors appear to be remarkably few. It would, for instance, be conceivable that the fundamental invariant interval is the fourth root of some in-invariant expression which is a quartic function of the co-ordinate differences. E.g.,

$$*R_{\alpha\beta\eta}{}^{\theta}\,*R_{\gamma\epsilon\theta}{}^{\eta}A^{\alpha}A^{\beta}A^{\gamma}A^{\epsilon}$$

might be associated with a displacement A^{λ}. Such a complication, however, would appear to be purely gratuitous, involving no clear gain on the physical side. There is just the one in-invariant of a simple character available, viz.,

$$^*G_{\alpha\beta}A^{\alpha}A^{\beta}.$$

We can construct a number of "densities" whose weights are zero, and which, when multiplied by the element of four-dimensional volume, are invariants; e.g.,

$$q\,^*G_{\alpha\beta}\,^*G^{\alpha\beta},\ q\,^*R_{\alpha\beta\gamma}{}^{\epsilon}\,^*R^{\alpha\beta\gamma}{}_{\epsilon},\ qK_{\alpha\beta}K^{\alpha\beta},\ q(^*G)^2.$$

Invariant densities of weight zero based on a sextic fundamental tensor are known.

The number of distinct characteristics of the world expressible by these absolute invariants, which are independent of co-ordinate and gauge-system, are remarkably few. There is one striking fact, first pointed out by Weyl. Only in a continuum of *even* dimensions can we have absolute invariant densities. The factor q or $\sqrt{(-g)}$ has a weight $n/2$ in a continuum of n dimensions. If n were odd, this weight would involve the fraction $\frac{1}{2}$, which could not be counteracted by the necessarily *integral* weight of any tensor.

If we decide to develop our geometry in a no more complicated fashion than is required by the physical facts, there are only available the very tensors which have been used.

The physical science of to-day is confronted with two difficulties, which may possibly turn out to be two aspects of the one problem. One has already been referred to; it is the impossibility of deducing the existence of the forces which hold the parts of an electron together from the Maxwell equations, however developed and amplified. The other concerns the fact, experimentally verified over a wide range of phenomena, that there exists an undoubted atomicity in the quantity which we call "action." The various formulæ involving Planck's quantum constant h, whose validity has been firmly established by numerous experimental researches in radiation, photo-electricity, atomic heats, X-rays, can all be deduced from a proposition of statistical mechanics; but that proposition itself cannot be deduced from the application of the *classical* principles of dynamics and the theory of probability. Now, as these principles have given way to the wider dynamics of Relativity, it is not unnatural to entertain the hope that in the development of the Relativity theory we might find some

justification for the introduction of a natural unit of action. Unfortunately, neither difficulty shows any sign of being resolved at the moment; but a few words may be written as a conclusion to this volume, concerning the suggestions which have been made as to possible modes of attack.

In the first place, if we regard electrons and protons as structures made up of parts, it is a "miracle," to quote Eddington's picturesque phrase, for an electron or proton to exist at all. We need some kind of external field (such as the Poincaré pressure) to keep it from "exploding," yet we cannot find any evidence for the existence of such a field in our equations. We can, of course, introduce this field as an empirical principle into our body of equations, just as Planck introduced the hypothesis of finite regions of equal probability into statistical mechanics to get round the experimental discrepancies which arose in radiation, specific heat, etc. But we are not happy with these strange neighbours. The "miracle" is very obvious in the case of an electron at rest; but, of course, if Relativity is true, it is equally a miracle for an electron in motion. But apart from this broad application of the principle, the miraculousness for the moving electron can be established by an analysis which is instructive. Thus in an impressed external field an electron will move with an accelerated motion, and will itself produce a field of force. It is conceivable that this field, due to the electron's motion, might be such as not only to equilibrate within the electron the external field producing the motion, but, in addition, to supply a component on each part of the electronic structure sufficient to equilibrate the disintegrating forces of the electron as a whole on this part. That this is not so can be established in detail as follows.

For a slowly moving element δe of charge, the electromagnetic potential at a point distant r from the element is

$$v_x\delta e/r,\ v_y\delta e/r,\ v_z\delta e/r,\ \delta e/r,$$

provided r is so small that we need not concern ourselves about retarded values. To generalise this to a form compatible with Relativity is not difficult. The components are

$$(\delta e/r)_0 V^\mu,$$

where V^μ is the vector $\dot{x}_\mu$ or dx_μ/ds and $(\delta e/r)_0$ is the value as calculated for Galilean axes in which the electron is momentarily at rest, so that we regard $(\delta e/r)_0$ as an invariant. With this understanding we can drop the zero suffix. Considering

that all parts of the electron have the same velocity or V^μ, the potential due to the electron is

$$V^\mu \int de/r,$$

integrating throughout the electron. But this is a contravariant vector, and we must employ a covariant vector in determining the field due to the electron's motion. This is A_μ where

$$\begin{aligned} A_\mu &= -g_{\mu\alpha} V^\alpha \int de/r \\ &= -V_\mu \int de/r. \end{aligned}$$

The minus sign is necessary, since in Galilean axes

$$g_{11} = g_{22} = g_{33} = -1 \text{ and } g_{44} = 1.$$

The force tensor due to the electron is thus

$$\begin{aligned} &(A_\nu)_\mu - (A_\mu)_\nu \\ &= [(V_\mu)_\nu - (V_\nu)_\mu] \int de/r. \end{aligned}$$

Multiplying the value of this at each point of the electron by the element of the electron's charge there, we find the force on the electron due to its own movement is

$$\begin{aligned} &[(V_\mu)_\nu - (V_\nu)_\mu] \iint de_1 de_2/r_{12} \\ &= [(V_\mu)_\nu - (V_\nu)_\mu] e^2/a, \end{aligned}$$

where $1/a$ is an average value of $1/r_{12}$ over every pair of points in the electron.

Now if the *external* field tensor is $F_{\mu\nu}$, the *covariant* force vector on the electron is $F_{\mu\alpha}J^\alpha$ where

$$J^\mu = eV^\mu.$$

The actual equation of motion is, of course, given by

$$m_0(\ddot{x}_\mu + \{\alpha\beta, \mu\}\dot{x}_\alpha \dot{x}_\beta) = -g^{\mu\alpha} F_{\alpha\beta} J^\beta,$$

where m_0 is the proper inertial mass of the electron. (See Chapter X., page 223.)

But

$$\ddot{x}_\mu + \{\alpha\beta, \mu\}\dot{x}_\alpha \dot{x}_\beta = V^\alpha (V^\mu)_\alpha.$$

Hence

$$F^\mu{}_\alpha J^\alpha = -m_0 V^\alpha (V^\mu)_\alpha.$$

Therefore

$$\begin{aligned} F_{\mu\alpha} J^\alpha &= -m_0 V^\alpha (V_\mu)_\alpha \\ &= -m_0 V^\alpha [(V_\mu)_\alpha - (V_\alpha)_\mu] \end{aligned}$$

because

$$V^{\alpha}(V_{\alpha})_{\mu} = 0.^{*}$$

As
$$J^{\mu} = eV^{\mu},$$
it follows that
$$eF_{\mu\nu} = -m_0[(V_{\mu})_{\nu} - (V_{\nu})_{\mu}].$$

In consequence, if
$$m_0 = e^2/a,$$
the external force on the electron $eF_{\mu\nu}$ is just equal and opposite to the force on the electron due to the motion produced by the external field. The electron's motion just neutralises the external field in the electron itself. There remains nothing left to equilibrate the force on the electron due to the mutual repulsion of its parts.

We cannot find within the older body of theory any hope of deducing the existence of the cohesive forces. Added to this, there is another feature to which reference has been previously made. The invariant T of the matter tensor $T_{\mu}{}^{\nu}$ is certainly not zero; it is the (averaged) proper density of the matter. But the invariant E of the electromagnetic energy tensor $E_{\mu}{}^{\nu}$ is zero, showing that it is impossible to build up electrons, and therefore matter, from electromagnetic fields alone. Some other form of energy must be present, the energy of the non-Maxwellian forces or the Poincaré pressure. Is there any hope of finding its formal expression by applying the identification method to some of the tensors obtained in the previous parts of this chapter, which have not had some physical counterpart assigned to them? The work of Weyl and Eddington provides us with absolute tensor and invariant curvatures which, when analysed, fall into parts some of which can be identified with gravitational and electromagnetic tensors. There still remain geometrical tensors such as C_{μ} and $H_{\mu\nu}$ unidentified with any physical tensor.

From (21) we have

$$^{*}G_{\mu\nu} = G_{\mu\nu} + H_{\mu\nu} + K_{\mu\nu}.$$

Also, referring to (22), we see that

$$H_{\mu\nu} = U_{\mu\alpha}{}^{\beta}U_{\nu\beta}{}^{\alpha} + 2U_{\mu\nu}{}^{\alpha}B_{\alpha} - [(U_{\mu\nu}{}^{\alpha})_{\alpha} + (B_{\mu})_{\nu} + (B_{\nu})_{\mu}], \quad (33)$$

and by contraction of $H^{\mu}{}_{\nu}$ we obtain

$$H = U_{\alpha\beta\gamma}U^{\alpha\gamma\beta} + 2B_{\alpha}C^{\alpha} - [2(B^{\alpha})_{\alpha} + (C^{\alpha})_{\alpha}]. \quad . \quad (34)$$

* $$V^{\alpha}V_{\alpha} = 1.$$
Hence $$V_{\alpha}(V^{\alpha})_{\mu} + V^{\alpha}(V_{\alpha})_{\mu} = 0$$
and it is easy to see that $V_{\alpha}(V^{\alpha})_{\mu} = V^{\alpha}(V_{\alpha})_{\mu}$.

Since
$$T_\mu{}^\nu = k[G_\mu{}^\nu - \tfrac{1}{2}g_\mu{}^\nu(G - 2\lambda)],$$
it follows that
$$T \;\; = k(4\lambda - G).$$
Now
$$I = g^{\alpha\beta}I_{\alpha\beta} = g^{\alpha\beta}\lambda g_{\alpha\beta} = 4\lambda.$$
Hence
$$\left.\begin{aligned} T &= k(I - G) \\ &= kH \end{aligned}\right\} \quad . \quad . \quad . \quad (35)$$
Also
$$\left.\begin{aligned} T_\mu{}^\nu &= k[G_\mu{}^\nu - \tfrac{1}{2}g_\mu{}^\nu(G - 4\lambda) - g_\mu{}^\nu\lambda] \\ &= k[G_\mu{}^\nu - \tfrac{1}{2}g_\mu{}^\nu(G - I) - I_\mu{}^\nu] \\ &= -k[H_\mu{}^\nu - \tfrac{1}{2}g_\mu{}^\nu H] \end{aligned}\right\} \quad . \quad (36)$$

Equation (36) thus gives us another expression for the complete matter tensor in terms of a geometric tensor, which is the remaining part of the absolute curvature tensor after removing the gravitational tensor and the electromagnetic tensor. Just as equation (31) gives the gravitational aspect of material energy, so equation (36) will give its electrical aspect, for it will be remembered that $U_{\lambda\mu}{}^\nu$, the tensor from whose components $H_{\mu\nu}$ is formed, is $f_{\lambda\mu}{}^\nu - \{\lambda\mu, \nu\}$, and Weyl and Eddington's theory claims this differs from zero only in regions where electrical energy is present. It is presumed that in empty space the curvature of the world, using an approximately Galilean frame, is 10^{-50} (cm.)$^{-2}$, and $U_{\lambda\mu}{}^\nu$ is of this order of magnitude except in electric fields so intense as to alter the local curvature enormously. In electromagnetic fields external to the electron the first two terms in (33) and (34), being product terms, will contribute practically nothing to $H_\mu{}^\nu$ and H. But when the electric field is very intense, as we may assume that it is within an electron, the product terms will be the preponderant part. Indeed, for an electron at rest we know by the identification of B_λ with the electromagnetic potential that $(B^\alpha)_\alpha$ is zero. We have no physical analogue to fit to C_λ, but it may reasonably vanish also in the static case (it does in Weyl's limitation). Hence the mass of an isolated electron is, from the electrical aspect, represented in all probability by the product terms in the expression (34) for H. In connection with this, it may be as well to remind the reader of the elementary point that in calculating the "electromagnetic inertia" of an electron in the usual manner, what we really do is to find a mathematical expression for the energy of the magnetic field due to its motion. It is
$$ne^2a^{-1}f(v)v^2$$

where n is a small numerical factor depending on units, $f(v)$ is a function of the velocity which approaches unity as v approaches zero, and a is the radius of a sphere *within which we suppose, the field not to exist* or on the usual law of force we would obtain an infinite result.

In short, we find in this way the *added* energy due to the electron's motion, and derive the rest-mass by drawing an analogy between that and mechanical kinetic energy. But we do not in this way obtain directly the rest-energy of the electron. We have seen already in the previous chapter that ordinary electrical theory will not supply it all even from the hypothesis of a static surface charge of e on a sphere of radius a. None of the familiar concepts of the Maxwell theory will supply the necessary material for the problem. There remains the hope, as yet unattained, that from the tensor $H_{\mu\nu}$ we may derive the physical theory which will fill the gap and provide us with the conditions of stability for the electronic structure as well as with the missing energy.

It might be thought that the tensors $T_\mu{}^\nu$ and $E_\mu{}^\nu$, given by

$$\begin{aligned} T_\mu{}^\nu &= -k[H_\mu{}^\nu - \tfrac{1}{2}g_\mu{}^\nu H] \\ E_\mu{}^\nu &= F_{\mu\alpha}F^{\nu\alpha} - \tfrac{1}{4}g_\mu{}^\nu F_{\alpha\beta}F^{\alpha\beta} \\ &= f^2(K_{\mu\alpha}K^{\nu\alpha} - \tfrac{1}{4}g_\mu{}^\nu K_{\alpha\beta}K^{\alpha\beta}) \end{aligned}$$

could be brought into direct relation by means of the third order tensor $U_{\lambda\mu}{}^\nu$ or $f_{\lambda\mu}{}^\nu - \{\lambda\mu, \nu\}$ (since $H_{\mu\nu}$ and $K_{\mu\nu}$ are both derived from $U_{\lambda\mu}{}^\nu$ by defined processes), and in consequence a direct expression for the non-Maxwellian energy tensor $T_\mu{}^\nu - E_\mu{}^\nu$ obtained. However, the mathematical difficulties are considerable, and have not so far been surmounted.

Eddington has arrived at an interesting conclusion concerning the natural units in which physical quantities are measured. This he does by applying the method of Hamiltonian variation to the invariant integral

$$\int q^*G_{\alpha\beta}{}^*G^{\beta\alpha}d\omega,$$

which is not altered by any change in the *natural* gauge, i.e. it is independent of the number chosen to represent λ, and so is a pure number. (It is, of course, altered by any arbitrary change of gauges, in which the ψ function is not a constant.) The following is a modification of Eddington's analysis, which dispenses with the variation process, and bases itself on a hypothesis which is admittedly very tentative, but yet has an element of plausibility to recommend it.

Consider what we may call a "world" gauge-co-ordinate system, i.e. one in which the unit gauge is the radius of the world, and in which observers would regard the local part of the co-ordinate mesh as equivalent to a Cartesian co-ordinate system in which two points on the same axis 1 cm. apart would have a δx_λ equal to ρ^{-1}, where the world radius is ρ cms.; briefly, the unit-mesh is estimated locally to have a width equal to the world radius. Now we have been identifying certain physical tensors with geometrical tensors which arise in the discussion of the curvature of the continuum which represents space-time. It is only natural to look for a relation between the units in which we measure physical quantities and the unit of curvature used in the measurement of the geometrical tensors; or, what comes to the same thing, the unit of length used in the geometrical quantities. It is also plausible to regard the unit of a physical quantity which is directly connected with the length unit in which the world radius is unity as being a natural unit of the quantity.

Now in the gauge-co-ordinate system referred to

$$I_{\mu\nu} = g_{\mu\nu},$$

which in order to keep dimensions correct should really be written

$$I_{\mu\nu} = g_{\mu\nu} \times \text{unit of curvature.}$$

From $g_{\mu\nu}$ we construct as usual the indexes $\{\lambda\mu, \nu\}$, and thence the Riemannian curvature tensor $R_{\lambda\mu\nu}{}^{\kappa}$ and the Einstein tensor $G_{\mu\nu}$. A subtraction of $G_{\mu\nu}$ from $I_{\mu\nu}$ gives $H_{\mu\nu}$. It is, of course, impossible to equate the parts into which $*G_{\mu\nu}$ is resolved, viz., $I_{\mu\nu}$ and $K_{\mu\nu}$, to each other anywhere, since they are tensors of different character, one being symmetric and one anti-symmetric. But we are going to equate

$$-I_{\mu\alpha}G^{\nu\alpha} + \tfrac{1}{4}g_{\mu}{}^{\nu}I_{\alpha\beta}G^{\alpha\beta}$$

to

$$K_{\mu\alpha}K^{\nu\alpha} - \tfrac{1}{4}g_{\mu}{}^{\nu}K_{\alpha\beta}K^{\alpha\beta}$$

in space unoccupied by electrons. There are no purely mathematical objections, since both expressions are mixed tensors; but the reasons for the procedure are physical. Reference to the gauging equation and to the considerations introduced in Chapter XV., in the paragraphs immediately succeeding equation (9), shows that the first of these expressions is the matter-tensor expressed in some as yet unknown unit. The second is the electromagnetic energy-tensor in the same unit.

Physically, we know that these tensors are equal in an electromagnetic field free of electrons. Hence, after an obvious modification of the first expression,

$$-[G_\mu^\nu - \tfrac{1}{2}g_\mu^\nu(G-2)] = K_{\mu\alpha}K^{\nu\alpha} - \tfrac{1}{4}g_\mu^\nu K_{\alpha\beta}K^{\alpha\beta}$$

in a purely electromagnetic field.

Let us transform gauge and co-ordinates to a centimetre gauge and a mesh whose unit is estimated locally to be 1 centimetre wide, so that indicating the transformed quantities by accented symbols, we have

$$\delta x_\mu = \rho^{-1}\delta x_\mu' = \lambda^{\frac{1}{2}}\delta x_\mu'$$

where λ cm.$^{-2}$ is the curvature of the world. Since $I_{\alpha\beta}\delta x_\alpha \delta x_\beta$, and $K_{\alpha\beta}\delta x_\alpha \delta x_\beta$ are in-invariants, it follows that

$$I_{\mu\nu} = \lambda^{-1}I_{\mu\nu}'$$
$$K_{\mu\nu} = \lambda^{-1}K_{\mu\nu}'.$$

Moreover, as we have seen a little earlier,

$$I_{\mu\nu}' = \lambda g_{\mu\nu}'$$

and therefore $g_{\mu\nu} = g_{\mu\nu}'$.

(This could of course be directly deduced from the equation

$$g_{\alpha\beta}\delta x_\alpha \delta x_\beta = \lambda g_{\alpha\beta}'\delta x_\alpha'\delta x_\beta'.)$$

It follows at once that

$$G_{\mu\nu} = \lambda^{-1}G_{\mu\nu}'\,;\ \ G_\mu^\nu = \lambda^{-1}G'^\nu_\mu$$
$$K^{\mu\nu} = \lambda^{-1}K^{\mu\nu\prime}.$$

Hence in a purely electromagnetic field

$$-\lambda^{-1}[G'^\nu_\mu - \tfrac{1}{2}g'^\nu_\mu(G'-2\lambda)] = K_{\mu\alpha}K^{\nu\alpha} - \tfrac{1}{4}g_\mu^\nu K_{\alpha\beta}K^{\alpha\beta}.$$

But we know from the problem dealt with in Chapter XI. that the left-hand expression is equal to

$$8\pi\kappa\lambda^{-1}T'^\nu_\mu$$

where T'^ν_μ is the matter-tensor expressed in such a unit that its mass component is T'^4_4 grams per cubic centimetre. So if E_μ^ν is the electromagnetic energy-tensor *expressed in the natural unit*, our result shows that the matter-tensor in the electromagnetic field *when expressed in gram units* is

$$\lambda(8\pi\kappa)^{-1}E_\mu^\nu\,;$$

or, in other words, the energy-density of the field is $\lambda(8\pi\kappa)^{-1}E_4^4$

grams per c.c., or $c^2\lambda(8\pi\kappa)^{-1}E_4{}^4$ ergs per c.c. Now the natural unit electromagnetic field will be one in which twice $E_4{}^4$ is unity, i.e. one with an energy density of

$$c^2\lambda/16\pi\kappa$$

ergs per c.c. Taking λ to be 10^{-50} this works out to be approximately ·0025 ergs per c.c. An electrostatic field with such an energy-density would have a field intensity about $\frac{1}{4}$ in electrostatic units, or 75 volts per cm.

If we take the world radius, about 10^{25} cm., as a natural unit of length, we have a natural unit of time, viz., $10^{15}/3$ seconds the time it takes light to travel along it. Taking the radius of an electron as $\cdot 6 \times 10^{-13}$ cm., we can find the amount of energy within a volume of the natural unit field equal to the electron's volume; it works out to

$$3 \times 10^{-42} \text{ ergs.}$$

If we multiply this by the natural unit of time, we obtain approximately

$$10^{-27} \text{ erg-sec.}$$

It is a dangerous thing to lay much stress on numerical coincidences, but it is striking how near this calculation brings us to Planck's quantum of action.

Glancing back, we see that

$$\begin{aligned} {}^*G_{\mu\nu}{}' &= I_{\mu\nu}{}' + K_{\mu\nu}{}' \\ &= \lambda(g_{\mu\nu} + K_{\mu\nu}). \end{aligned}$$

The result can be stated in general terms thus. Using a co-ordinate system whose mesh has a unit width equal to the gauge used, then the in-tensor ${}^*G_{\mu\nu}$ is given by

$${}^*G_{\mu\nu} = \lambda(g_{\mu\nu} + F_{\mu\nu}).$$

Here $g_{\mu\nu}$ is the metric tensor whose value is obtained just as in Einstein's theory. $F_{\mu\nu}$ is the field-tensor in the natural unit introduced above (about $\frac{1}{4}$ electrostatic unit for electric and $\frac{1}{4}$ gauss for magnetic field). Also, λ is the curvature of the world in terms of the gauge used.

The fact that Eddington's geometry is less restricted than Weyl's can be seen very simply by observing that, while Weyl abandons the invariance of the expression for the line element, he retains the assumption that the ratio of the magnitudes of two line elements drawn from the same point is not altered by parallel displacement of the elements. Eddington introduces

no such postulate; he bases his geometrical theory entirely on the conception of parallel displacement and the analytical expression for it,

$$\delta A^{\mu} = - f_{\alpha\beta}{}^{\mu} A^{\alpha} \delta x_{\beta}.$$

The forty quantities $f_{\lambda\mu}{}^{\nu}$ are, at the outset, undetermined functions of the co-ordinates x_{μ}. He succeeds in arriving at a quantitative measure of anything to which the term "structure" can be applied, "structure" being considered as quite distinct from "substance." This measure is $^{*}G_{\mu\nu}$, and it resolves itself naturally into two parts, a symmetrical $I_{\mu\nu}$ and an anti-symmetrical $K_{\mu\nu}$. It is at this point that physical theory makes its appearance. Einstein, for example, builds up separation, and hence space, time, geometry, gravitation, and mechanics from $I_{\mu\nu}$; he derives Maxwell's theory of electrical phenomena from $K_{\mu\nu}$. He does so by recognising the physical manifestations of these two geometrical tensors. Thus his work is not an approximation; on the contrary, as far as Eddington's World geometry can lead to any conclusion, Einstein's postulates and theory are exact for the *natural* geometry of the World, the significance of the word "natural" emerging in the recognition of Einstein's law of gravitation as a definition of a physically convenient and unique gauge. Einstein's work is entirely unaffected by the ambiguity of length comparisons which enters at the initial stages of World Geometry; his separation is absolute, since it is an in-invariant, for the co-ordinate-gauge systems which he employs.

ADDENDUM.

There still remains, however, the master problem, viz., to discover the differential equations which determine the quantities $f_{\lambda\mu}{}^{\nu}$; just as the problem in the original Relativity Theory was to discover the differential equations satisfied by the fourteen coefficients, $g_{\mu\nu}$ and ϕ_{μ}. A paper * in which Einstein himself suggests a method for formulating such equations has just appeared (March, 1923), and a note giving an outline of this paper is added as the corrected proofs of these last pages are going to press.

* "Sitz. d. Preuss. Akad.," 1923, pp. 32-37.

The method is based on Hamilton's Principle of varying an Action integral, which was studied in Chapter XII. We observed in that connection that such an integral must be an invariant in order to satisfy the Principle of Relativity, and so the integrand is a scalar-density. In order to avoid digression later, let us deal with one or two mathematical points in connection with scalar and tensor densities. We shall denote such quantities by heavy Clarendon type.

First of all, if $A_{\lambda\mu}$ is *any arbitrary* covariant tensor,

$$A_{\alpha\beta}dx_\alpha dx_\beta$$

is invariant, and we can prove by the method adopted in Chapter VIII. in connection with the fundamental tensor of the original theory, that

$$|A|^{\frac{1}{2}}d\omega$$

is an invariant, where $|A|$ represents the determinant formed by the constituents of $A_{\mu\nu}$. Thus $|A|^{\frac{1}{2}}$ is a scalar-density, and the product of any tensor by $|A|^{\frac{1}{2}}$ is a tensor-density.

Further, owing to the invariance of $A_{\alpha\beta}dx_\alpha dx_\beta$, we can easily adapt the reasoning on page 359 of this chapter, to demonstrate that

$$\partial A_{\lambda\mu}/\partial x_\nu - f_{\mu\nu}{}^\alpha A_{\lambda\alpha} - f_{\lambda\nu}{}^\alpha A_{\mu\alpha}$$

is a covariant tensor of the third order. Hence it follows just as before that

$$f_{\lambda\mu}{}^\nu - \tfrac{1}{2}A^{\nu\alpha}(\partial A_{\mu\alpha}/\partial x_\lambda + \partial A_{\lambda\alpha}/\partial x_\mu - \partial A_{\lambda\mu}/\partial x_\alpha) \quad . \quad (37)$$

is a mixed tensor of the third order, where $A^{\mu\nu}$ is equal to the cofactor of $A_{\mu\nu}$ in $|A|$ divided by $|A|$.

Now in covariant derivation we must, of course, employ $f_{\lambda\mu}{}^\nu$ where we employed $\{\lambda\mu, \nu\}$ in the original theory, which was based on the invariance of the expression for the line-element as a fundamental postulate—a postulate now dropped in the initial stages of the theory. E.g., the covariant derivative of a contravariant tensor $B^{\lambda\mu}$ is now

$$\partial B^{\lambda\mu}/\partial x_\nu + f_{\alpha\nu}{}^\lambda B^{\alpha\mu} + f_{\alpha\nu}{}^\mu B^{\lambda\alpha},$$

and we shall use the symbol $(B^{\lambda\mu})_\nu$ to refer to it.

Similarly, we can speak of the covariant derivative of a tensor-density, if we can by some form of differential operator obtain a tensor-density from a tensor-density. It is not difficult to discover such an operator. Thus let $\mathbf{B}^{\lambda\mu}$ be a contravariant tensor-density, then we shall prove that

$$(\mathbf{B}^{\lambda\mu})_\nu = \partial\mathbf{B}^{\lambda\mu}/\partial x_\nu + f_{a\nu}{}^{\lambda}\mathbf{B}^{a\mu} + f_{a\nu}{}^{\mu}\mathbf{B}^{\lambda a} - f_{\nu a}{}^{a}\mathbf{B}^{\lambda\mu} \qquad (38)$$

is a tensor-density.

Putting $\mathbf{B}^{\lambda\mu} = |A|^{\frac{1}{2}}B^{\lambda\mu}$, it is easily seen that

$$(\mathbf{B}^{\lambda\mu})_\nu = |A|^{\frac{1}{2}}(B^{\lambda\mu})_\nu + \mathbf{B}^{\lambda\mu}[f_{\nu a}{}^{a} - |A|^{\frac{1}{2}}\partial|A|^{\frac{1}{2}}/\partial x_\nu].$$

Now a reference to pages 196, 197 of Chapter IX. will show that

$$|A|^{-\frac{1}{2}}\partial|A|^{\frac{1}{2}}/\partial x_\nu = \{\nu a, a\} \text{ (with } A \text{ written for } g\text{)},$$

and it will be at once apparent from (37) that the coefficient of $\mathbf{B}^{\lambda\mu}$ in the second term on the right-hand side is a covariant vector. Hence this second term is a mixed tensor-density, and as the first term is also a tensor-density, the fact is established for $(\mathbf{B}^{\lambda\mu})_\nu$.

Having disposed of this question of covariant differentiation, we can follow Einstein's proposed method for establishing fundamental equations in a new generalisation of the theory.

The postulate of parallel displacement leads to the Riemann tensor

$$R_{\lambda\mu\nu}{}^{\kappa} = \partial f_{\lambda\nu}{}^{\kappa}/\partial x_\mu - \partial f_{\lambda\mu}{}^{\kappa}/\partial x_\nu + f_{\lambda\nu}{}^{a}f_{a\mu}{}^{\kappa} - f_{\lambda\mu}{}^{a}f_{a\nu}{}^{\kappa},$$

and the contracted tensor

$$R_{\lambda\mu} = \partial f_{\lambda a}{}^{a}/\partial x_\mu - \partial f_{\lambda\mu}{}^{a}/\partial x_a + f_{\lambda a}{}^{\beta}f_{\mu\beta}{}^{a} - f_{\lambda\mu}{}^{a}f_{a\beta}{}^{\beta},$$

where we drop the asterisk, and write $R_{\lambda\mu}$ for $*G_{\lambda\mu}$.

Now write

$$R_{\mu\nu} = \lambda(g_{\mu\nu} + F_{\mu\nu})$$

where $g_{\mu\nu}$ is a symmetric tensor, $F_{\mu\nu}$ an anti-symmetric, and λ the curvature of the universe expressed in terms of our usual standards of length, so that

$$\left.\begin{aligned} \lambda g_{\mu\nu} &= \tfrac{1}{2}(\partial f_{\mu a}{}^{a}/\partial x_\nu + \partial f_{\nu a}{}^{a}/\partial x_\mu) - \partial f_{\mu\nu}{}^{a}/\partial x_a + f_{\mu a}{}^{\beta}f_{\nu\beta}{}^{a} \\ &\qquad - f_{\mu\nu}{}^{a}f_{a\beta}{}^{\beta} \\ \lambda F_{\mu\nu} &= \tfrac{1}{2}(\partial f_{\mu a}{}^{a}/\partial x_\nu - \partial f_{\nu a}{}^{a}/\partial x_\mu) \end{aligned}\right\} \qquad (39)$$

Now let $\mathbf{L}$ be a scalar-density which is a function of the $g_{\mu\nu}$ and $F_{\mu\nu}$, and let us write

$$\left.\begin{aligned} \partial\mathbf{L}/\partial g_{\mu\nu} &= \mathbf{A}^{\mu\nu} \\ \partial\mathbf{L}/\partial F_{\mu\nu} &= \mathbf{F}^{\mu\nu} \end{aligned}\right\} \quad . \quad . \quad . \quad (40)$$

If any arbitrary variations be imposed on the $g_{\mu\nu}$ and $F_{\mu\nu}$,

$$\delta\mathbf{L} = \mathbf{A}^{a\beta}\delta g_{a\beta} + \mathbf{F}^{a\beta}\delta F_{a\beta} \quad . \quad . \quad . \quad (41)$$

and since $\delta\mathbf{L}$ is a scalar-density, therefore $\mathbf{A}^{\mu\nu}$ and $\mathbf{F}^{\mu\nu}$ are

contravariant tensor-densities, symmetric and anti-symmetric respectively.

Now $\mathbf{A}^{\mu\nu}$ can be written in the form

$$\mathbf{A}^{\mu\nu} = (-\ |A|)^{\frac{1}{2}} A^{\mu\nu} \quad . \quad . \quad . \quad (42)^*$$

where $A_{\mu\nu}$ is a suitably chosen symmetrical covariant tensor, and $A^{\mu\nu}$ is the contravariant tensor obtained by dividing the cofactor of $A_{\mu\nu}$ in $|A|$ by $|A|$, so that

$$\begin{aligned} A_\mu^{\ \nu} &= A_{\mu\alpha} A^{\nu\alpha} \\ &= 1 \text{ if } \mu = \nu \\ &= 0 \text{ if } \mu \neq \nu \end{aligned}$$

and $$A_\alpha^{\ \alpha} = 4.$$

Also let $F^{\mu\nu}$ be an anti-symmetric covariant tensor such that

$$\mathbf{F}^{\mu\nu} = (-\ |A|)^{\frac{1}{2}} F^{\mu\nu}. \quad . \quad . \quad . \quad (43)$$

Since $\mathbf{F}^{\mu\nu}$ is anti-symmetric we can establish, just as in page 199, that

$$\partial \mathbf{F}^{\mu\alpha} / \partial x_\alpha$$

is a covariant vector-density, or

$$\begin{aligned} \partial \mathbf{F}^{\mu\alpha} / \partial x_\alpha &= \mathbf{J}^\mu \\ &= (-\ |A|)^{\frac{1}{2}} J^\mu \quad . \quad . \quad . \quad (44) \end{aligned}$$

where $\mathbf{J}^\mu$ is a vector-density and J^μ a vector.

Now consider $\mathbf{L}$ to be an action-density, or, in other words,

$$\int \mathbf{L} d\omega$$

to be an action function, which (presuming a suitable form is proposed for $\mathbf{L}$) will yield differential equations for the $f_{\lambda\mu}^{\ \nu}$ with a physical significance, when the process of variation is carried out for arbitrary variations of the $f_{\lambda\mu}^{\ \nu}$ and the $\partial f_{\lambda\mu}^{\ \nu} / \partial x_\kappa$. Equations (41) and (39) enable us to express the variation in terms of the $\delta f_{\lambda\mu}^{\ \nu}$ and the $\delta(\partial f_{\lambda\mu}^{\ \nu} / \partial x_\kappa)$. By the usual device of partial integration and the rejection of a boundary integral (on account of the postulated vanishing of the variations of the parameters at this boundary) we can express the variation in terms of the $\delta f_{\lambda\mu}^{\ \nu}$ alone. After some steps which are straightforward though tedious, it can be established that

$$\begin{aligned} \delta \int \mathbf{L} d\omega = \int d\omega \delta f_{\beta\gamma}^{\ \epsilon} \{ (\mathbf{A}^{\beta\gamma})_\epsilon - \tfrac{1}{2} A_\epsilon^{\ \beta} (\mathbf{A}^{\gamma\alpha})_\alpha - \tfrac{1}{2} A_\epsilon^{\ \gamma} (\mathbf{A}^{\beta\alpha})_\alpha \\ + (\mathbf{F}^{\beta\gamma})_\epsilon - \tfrac{1}{2} A_\epsilon^{\ \beta} (\mathbf{F}^{\gamma\alpha})_\alpha - \tfrac{1}{2} A_\epsilon^{\ \gamma} (\mathbf{F}^{\beta\alpha})_\alpha \}. \end{aligned}$$

* The minus sign under the root will be justified presently.

Putting the variation equal to zero, using (44), and remembering that

$$(\mathbf{F}^{\beta\gamma})_\epsilon \delta f_{\beta\gamma}{}^\epsilon = 0$$

on account of the anti-symmetry of $\mathbf{F}^{\beta\gamma}$, we obtain

$$(\mathbf{A}^{\lambda\mu})_\nu - \tfrac{1}{2}A_\nu{}^\lambda(\mathbf{A}^{\mu\alpha})_\alpha - \tfrac{1}{2}A_\nu{}^\mu(\mathbf{A}^{\lambda\alpha})_\alpha - \tfrac{1}{2}A_\nu{}^\lambda\mathbf{J}^\mu - \tfrac{1}{2}A_\nu{}^\mu\mathbf{J}^\lambda = 0. \quad (45)$$

Contracting for μ and ν, we obtain

$$3(\mathbf{A}^{\lambda\alpha})_\alpha + 5\mathbf{J}^\lambda = 0. \quad . \quad . \quad . \quad (46)$$

Combining (45) and (46), we finally arrive at

$$(\mathbf{A}^{\lambda\mu})_\nu + \tfrac{1}{3}A_\nu{}^\lambda\mathbf{J}^\mu + \tfrac{1}{3}A_\nu{}^\mu\mathbf{J}^\lambda = 0. \quad . \quad . \quad (47)$$

A glance at (40), (44), and (39) will show that the forty equations (47) are differential equations for the forty quantities $f_{\lambda\mu}{}^\nu$, involving the first and second differential coefficients with respect to the x_μ provided, of course, a functional form has been ascribed to $\mathbf{L}$. A solution of these equations would then yield by means of (39) values for $g_{\mu\nu}$ and $F_{\mu\nu}$, and we would proceed as before to identify the invariant

$$g_{\alpha\beta}\delta x_\alpha \delta x_\beta$$

as the square of the line element in the World, and the tensor $F_{\mu\nu}$ as the covariant tensor of the electromagnetic field, the contravariant being $F^{\mu\nu}$.

The question now arises as to a suitable form for $\mathbf{L}$. If, for instance, we put

$$\mathbf{L} = (-g)^{\frac{1}{2}},$$

and therefore, by (6) of Chapter IX.,

$$\mathbf{A}^{\mu\nu} = (-g)^{\frac{1}{2}}g^{\mu\nu},$$

and also

$$\mathbf{F}^{\mu\nu} = 0$$
$$\mathbf{J}^\mu = 0,$$

it can be easily derived from (46) and (47) that

$$f_{\lambda\mu}{}^\nu = \{\lambda\mu, \nu\}.$$

Equations (39) then become

$$\lambda g_{\mu\nu} = G_{\mu\nu}$$

and

$$F_{\mu\nu} = 0$$

and we have the original gravitational equations in a complete vacuum.

Guided by this result, Einstein now proposes to write

$$\mathbf{L} = 2(-|B|)^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad (48)$$

where

$$B_{\mu\nu} = g_{\mu\nu} + F_{\mu\nu}$$

and $|B|$ is the determinant of the $B_{\mu\nu}$-coefficients.

This proposal would give explicit expressions for the $\mathbf{A}^{\mu\nu}$ and $\mathbf{J}^{\mu}$ in terms of the $g_{\mu\nu}$ and $F_{\mu\nu}$, and would thus give definite form to the differential equations (47). The equations are very complicated, and no solution even for a single centre is attempted in the paper, but, if solved, they would presumably yield the material for constructing gravitation, electricity, and matter. It is a striking feature of this new proposal of Einstein's that the two fields are ideally fused into one concept; for, whereas in former treatments of the Action Principle, the fields, the matter, and the electricity were represented by separate and independent terms in the Action function, Einstein's Action function cannot be so divided up. The mathematical treatment links all the physical processes up in a mathematical function depending on the one quantity, viz., curvature.

Einstein points out one result which may prove to be of extreme importance if the new proposal be ultimately justified. To see how it arises we must carry the analysis a little further than equation (47). We can write this equation by means of (38) as

$$\partial A^{\lambda\mu}/\partial x_\nu + f_{\alpha\nu}{}^{\lambda}A^{\alpha\mu} + f_{\alpha\nu}{}^{\mu}A^{\lambda\alpha} + (\{\nu\alpha, \alpha\} - f_{\nu\alpha}{}^{\alpha})A^{\lambda\mu} + \tfrac{1}{3}A_\nu{}^{\lambda}J^{\mu} + \tfrac{1}{3}A_\nu{}^{\mu}J^{\lambda} = 0,$$

(where in $\{\lambda\mu, \nu\}$ we now understand that A is written instead of g).

But

$$\partial A^{\mu\lambda}/\partial x_\nu = -A^{\lambda\alpha}A^{\mu\beta}\partial A_{\alpha\beta}/\partial x_\nu$$

(see equation (3), p. 196),

and
$$A^{\lambda\mu} = A^{\lambda\alpha}A^{\mu\beta}A_{\alpha\beta}.$$

Hence writing

$$f_{\lambda\mu\nu} = A_{\nu\alpha}f_{\lambda\mu}{}^{\alpha}$$
$$[\lambda\mu, \nu] = A_{\nu\alpha}\{\lambda\mu, \alpha\}$$
$$J_\mu = A_{\mu\alpha}J^{\alpha}$$

we can obtain after a little manipulation

$$\partial A_{\lambda\mu}/\partial x_\nu - f_{\lambda\nu\mu} - f_{\mu\nu\lambda} = A_{\lambda\mu}(\{\nu\alpha, \alpha\} - f_{\nu\alpha}{}^{\alpha}) + \tfrac{1}{3}A_{\lambda\nu}J_\mu + \tfrac{1}{3}A_{\mu\nu}J_\lambda.$$

Now by (37)

$$\{\nu\alpha, \alpha\} - f_{\nu\alpha}{}^{\alpha}$$

is a covariant vector. Make for the present the tentative assumption that

$$\{\nu\alpha, \alpha\} - f_{\nu\alpha}{}^{\alpha} = aJ_{\nu}$$

where a is a constant to be determined. Then

$$\begin{aligned} &\partial A_{\lambda\mu}/\partial x_{\nu} - f_{\lambda\nu\mu} - f_{\mu\nu\lambda} \\ &= aA_{\lambda\mu}J_{\nu} + \tfrac{1}{3}A_{\lambda\nu}J_{\mu} + \tfrac{1}{3}A_{\mu\nu}J_{\lambda}. \end{aligned}$$

Hence writing three such equations by cyclic interchange of λ, μ, and ν, and subtracting one of them from the sum of the other two we have

$$2[\lambda\mu, \nu] - 2f_{\lambda\mu\nu} = (\tfrac{2}{3} - a)A_{\lambda\mu}J_{\nu} + aA_{\mu\nu}J_{\lambda} + aA_{\lambda\nu}J_{\mu}.$$

Hence

$$2\{\lambda\mu, \nu\} - 2f_{\lambda\mu}{}^{\nu} = (\tfrac{2}{3} - a)A_{\lambda\mu}J^{\nu} + aA_{\mu}{}^{\nu}J_{\lambda} + aA_{\lambda}{}^{\nu}J_{\mu}.$$

Putting $\mu = \nu = \alpha$ and contracting,

$$2\{\lambda\alpha, \alpha\} - 2f_{\lambda\alpha}{}^{\alpha} = (\tfrac{2}{3} - a + 4a + a)J_{\lambda}.$$

Hence our tentative assumption is justified if a is put equal to $-\frac{1}{3}$, so that we obtain finally

$$f_{\lambda\mu}{}^{\nu} = \{\lambda\mu, \nu\} - \tfrac{1}{2}A_{\lambda\mu}J^{\nu} + \tfrac{1}{6}A_{\lambda}{}^{\nu}J_{\mu} + \tfrac{1}{6}A_{\mu}{}^{\nu}J_{\lambda} \quad . \quad (49)$$

and

$$f_{\lambda\alpha}{}^{\alpha} = \{\lambda\alpha, \alpha\} + \tfrac{1}{3}J_{\lambda}. \quad . \quad . \quad . \quad (50)$$

By (39) and (50) we see that

$$6\lambda F_{\mu\nu} = \partial J_{\mu}/\partial x_{\nu} - \partial J_{\nu}/\partial x_{\mu}.$$

It follows that the vector potential of the field is

$$J_{\mu}/6\lambda,$$

which shows that there must be charge and current at every part of the field, a result not contemplated in the earlier electromagnetic theory. Of course, for ordinary values of the field-tensor it is apparent that the values of J_{μ} will be minutely small on account of the excessive smallness of λ. Even for values of $F_{\mu\nu}$, which are supposed to exist at the surface of the electron on the usual theory (of the order 10^{16} in electrostatic units, and therefore also in our fundamental field unit), we find them quite compatible with small values of J_{μ}. Of course, the

proper vector to be considered in discussing density of charge or current is J^μ and

$$J^\mu = A^{\mu\alpha} J_\alpha.$$

But even though the $A^{\mu\nu}$ be large where the field is strong, it appears on consideration that the minuteness of λ still keeps the J^μ of reasonable magnitude for relatively enormous values of $F_{\mu\nu}$. The new theory, therefore, holds out hopes of constructing matter from electricity without the trouble of the singularities involved in the earlier electron theory. The theory, in fact, points to a general diffusion of " electricity " throughout the field ; but it will, to be successful, have to account for the existence of the magnitudes which we call electron charge, mass, and size. It is possible that insight into these problems will only come by absorbing the quantum hypothesis at some point into the general theory.

One other point occurs to the writer as he brings this hurried sketch of Einstein's latest formulation to a close. If a solution be obtained for equations (47) which is static, i.e. does not involve x_4, and in which, by analogy with the static solution for a single centre treated in Chapter XIII., $g_{\mu 4}$ and $F_{\mu 4}$ are zero where $\mu = 1, 2$, or 3, it follows that the resulting field will be gravitational and *magnetostatic*, for F_{23}, F_{31}, F_{12} represent magnetic components and F_{14}, F_{24}, F_{34} electric. This would appear to point to the conclusion that matter is ultimately built up of *magnetic* charges, a view already hinted at in Weyl and Eddington's writings.*

In conclusion, the reader will observe that the Relativity Theory has in all its successive generalisations, from the form of the theory as first published in 1905, proceeded by the modification of its mathematical method but not its point of view. In all cases the invariance of mathematical forms of the equations of Physics has been its unalterable feature. At first it involved as a subsidiary postulate the invariant value of the velocity of light, leading to the conclusion that this unique velocity is the limiting velocity for any moving element. In 1915 this is abandoned and replaced by the invariance of the line element

* See, for example, page 211 of Eddington's latest book, "The Mathematical Theory of Relativity."

in curved space-time, coupled with the subsidiary postulate that the invariant value is zero for the element of a light-track. There still remained two difficulties—the apparent absoluteness of rotation, and the objection which carried considerable weight with some physicists that gravitation occupied a privileged position in the theory of 1915 as compared with electromagnetic force, which physically was equally important. The first was met by the hypothesis of a spherical or spherical-cylindrical world. Weyl made the first step towards removing the second, and Eddington, in completing that step, starts from the existence of parallel displacement and not the invariance of a line element. Einstein now proposes a form for the new equations of "gravitation-cum-electromagnetism" required by this advance. It remains to be seen if some striking experimental verification will lend that strong support which was not wanting in the case of the earlier forms of the theory, but which has hitherto been denied to the conclusions arrived at since Weyl introduced his idea of ambiguity in gauging.

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